A short elementary proof of $\sum 1/k^2 = \pi^2/6$

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Abstract

We give a short elementary proof of the well known identity $\zeta(2) = \sum_{k=1}^{\infty} 1/k^2 = \pi^2/6$. The idea is to write the partial sums of the series as a telescoping sum and to estimate the error term. The proof is based on recursion relations between integrals obtained by integration by parts, and simple estimates.

Introduction

The aim of this note is to give a truly elementary proof of the identity

$$\zeta(2) = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \tag{1}$$

which can be appreciated by anyone who understands elementary calculus. The identity (1) is often referred to as the "Basel Problem" and was solved by Euler around 1735. More on the interesting history can be found in [5, 15].

The idea in this paper is to derive an explicit formula for the partial sums of (1) by rewriting it as a telescoping sum. For that we exploit recursion relations between the integrals

$$A_n = \int_0^{\frac{\pi}{2}} \cos^{2n} x \, dx \qquad \text{and} \qquad B_n = \int_0^{\frac{\pi}{2}} x^2 \cos^{2n} x \, dx$$

for $n \geq 0$. In particular we derive the explicit estimate

$$0 \leq \frac{\pi^2}{6} - \sum_{k=1}^{n} \frac{1}{k^2} = 2\frac{B_n}{A_n} \leq \frac{\pi^2}{4(n+1)} \tag{2}$$

from which (1) follows by letting $n \to \infty$. The idea is similar to the one by Masuoka [13], but the estimate of the remainder term is even simpler. An alternative way to write (1) as a telescoping sum is given in [2].

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There are many short proofs of (1), but most rely on additional knowledge. A nice collection is given in [3]. One proof commonly used is based on non-trivial theorems on the pointwise convergence of Fourier series. A second approach is based on the Euler-MacLaurin summation formula (see [6, Section II.10] or [4]). Other proofs rely on the product formula for $\sin x$ such a Euler's original proof (see [6, pp 62–67] or [5, 15]). Yet other proofs involve complex analysis such as the one in [12] or double integrals and Fubini's theorem [1, 7, 8, 10]. Without attempting to provide a complete list there are proofs in [4, 9, 11, 14] and references therein.

Derivation of the result

We start by proving the well known recursion relations between A_n and A_{n-1}. Using integration by parts and the identity $\sin^2 x = 1 - \cos^2 x$

$$\begin{aligned} A_n &= \int_0^{\frac{\pi}{2}} \cos x \cos^{2n-1} x\,dx = (2n-1)\int_0^{\frac{\pi}{2}} \sin^2 x \cos^{2(n-1)} x\,dx \\ &= (2n-1)\int_0^{\frac{\pi}{2}} (1-\cos^2 x)\cos^{2(n-1)} x\,dx = (2n-1)(A_{n-1}-A_n). \end{aligned}$$

Hence for $n \geq 1$

$$\int_0^{\frac{\pi}{2}} \sin^2 x \cos^{2(n-1)} x\,dx = \frac{A_n}{2n-1} = \frac{A_{n-1}}{2n}. \tag{3}$$

Next we rewrite A_n in terms of B_n and B_{n-1}. The idea is to use integration by parts twice, introducing the factors x, and then x^2. Using integration by parts a first time we get

$$A_n = \int_0^{\frac{\pi}{2}} 1 \times \cos^{2n} x\,dx = 2n\int_0^{\frac{\pi}{2}} x \sin x \cos^{2n-1} x\,dx.$$

Using integration by parts a second time we get

$$\begin{aligned} A_n &= -n\int_0^{\frac{\pi}{2}} x^2\big(\cos x\cos^{2n-1} x - (2n-1)\sin^2 x\cos^{2n-2} x\big)\,dx \\ &= -nB_n + n(2n-1)\int_0^{\frac{\pi}{2}} x^2(1-\cos^2 x)\cos^{2(n-1)}\,dx \\ &= (2n-1)nB_{n-1} - 2n^2 B_n. \end{aligned}$$

Hence for all $n \geq 1$ we have

$$A_n = (2n-1)nB_{n-1} - 2n^2 B_n. \tag{4}$$

This allows us to derive a simple expression for the partial sums of (1). Dividing the identity in (4) by $n^2 A_n$ and then using (3)

$$\frac{1}{n^2} = \frac{(2n-1)B_{n-1}}{nA_n} - \frac{2B_n}{A_n} = \frac{2B_{n-1}}{A_{n-1}} - \frac{2B_n}{A_n}$$

for all $n \geq 1$. Hence we have the telescoping sum

$$\sum_{k=1}^{n} \frac{1}{k^2} = \sum_{k=1}^{n} \Big(\frac{2B_{k-1}}{A_{k-1}} - \frac{2B_k}{A_k} \Big) = \frac{2B_0}{A_0} - \frac{2B_n}{A_n}$$

for all $n \geq 1$. Now

$$A_0 = \int_0^{\frac{\pi}{2}} 1\, dx = \frac{\pi}{2} \qquad \text{and} \qquad B_0 = \int_0^{\frac{\pi}{2}} x^2\, dx = \frac{\pi^3}{3 \times 8},$$

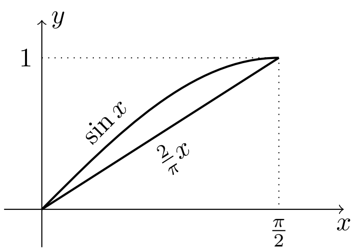
and so

$$\frac{2B_0}{A_0} = \frac{\pi^2}{6},$$

Hence for all $n \geq 1$ we have

$$\sum_{k=1}^{n} \frac{1}{k^2} = \frac{\pi^2}{6} - 2\frac{B_n}{A_n}. \tag{5}$$

We now estimate B_n in terms of A_n to get a bound for B_n/A_n. The linear function $2x/\pi$ coincides with $\sin x$ for $x = 0$ and for $x = \pi/2$. Because $\sin x$ is concave on $[0, \pi/2]$ we get $\sin x \geq 2x/\pi$ for all $x \in [0, \pi/2]$ as illustrated below.



Using the recursion relation (3) with n replaced by $n+1$ we get

$$B_n \;=\; \int_0^{\frac{\pi}{2}} x^2 \cos^{2n} x\, dx \;\leq\; \Big(\frac{\pi}{2}\Big)^2 \int_0^{\frac{\pi}{2}} \sin^2 x \cos^{2n} x\, dx \;=\; \frac{\pi^2}{4} \frac{A_n}{2(n+1)}.$$

Combining the above with (5) we arrive at (2) as required.

We finally note that an induction using (3) and (5) gives Masuoka's representation from [13], namely

$$\sum_{k=1}^{n-1} \frac{1}{k^2} = \frac{\pi^2}{6} - \frac{\pi}{4} \frac{(2n)!!}{(2n-1)!!} B_n,$$

but we have dealt with the error term rather more directly.

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