Krahn's proof of the Rayleigh conjecture revisited

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Abstract. The paper is a discussion of Krahn's proof of the Rayleigh conjecture that amongst all membranes of the same area and the same physical properties, the circular one has the lowest ground frequency. We show how his approach coincides with the modern techniques of geometric measure theory using the co-area formula. We furthermore discuss some issues and generalisations of his proof.

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1. Introduction

In his book *The Theory of Sound* [36] first published in 1877 Lord Rayleigh made a famous conjecture:

> *If the area of a membrane be given, there must evidently be some form of boundary for which the pitch (of the principal tone) is the gravest possible, and this form can be no other than the circle. . . .*

The conjecture is supported by a table of explicit values for the ground frequency of a membrane. We reproduce the table from the 1945 Dover edition [35] in Figure 1. On that table the circle clearly exhibits the lowest value. Rayleigh further provides a rather lengthy proof in case of a near circle using perturbation series involving Bessel functions.

The conjecture remained unproved for a very long time. Courant [11] obtained a related but weaker result, namely that the circle minimises the ground frequency amongst all membranes with the same circumference. In 1923 Faber published a proof in the *Sitzungsbericht der bayerischen Akademie der Wissenschaften* [17]. Independently, a proof by Krahn appeared in 1925 in *Mathematische Annalen* [25]. According to [28] the result was intended

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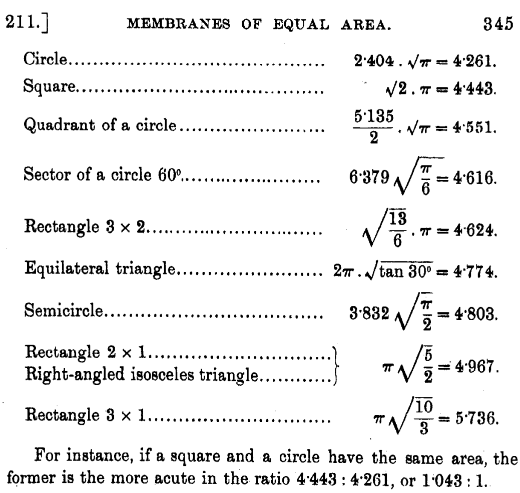

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Circle	$2{\cdot}404 \,.\, \sqrt{\pi} = 4{\cdot}261.$
Square	$\sqrt{2} \,.\, \pi = 4{\cdot}443.$
Quadrant of a circle	$\frac{5{\cdot}135}{2} \,.\, \sqrt{\pi} = 4{\cdot}551.$
Sector of a circle 60°	$6{\cdot}379 \sqrt{\frac{\pi}{6}} = 4{\cdot}616.$
Rectangle 3 × 2	$\sqrt{\frac{13}{6}} \,.\, \pi = 4{\cdot}624.$
Equilateral triangle	$2\pi \,.\, \sqrt{\tan 30^\circ} = 4{\cdot}774.$
Semicircle	$3{\cdot}832 \sqrt{\frac{\pi}{2}} = 4{\cdot}803.$
Rectangle 2 × 1 } Right-angled isosceles triangle }	$\pi \sqrt{\frac{5}{2}} = 4{\cdot}967.$
Rectangle 3 × 1	$\pi \sqrt{\frac{10}{3}} = 5{\cdot}736.$

For instance, if a square and a circle have the same area, the former is the more acute in the ratio 4·443 : 4·261, or 1·043 : 1.

FIGURE 1. Rayleigh's table showing explicit values for the ground frequency of membranes of various shapes

to be the doctoral thesis of Krahn at the University of Göttingen under the direction of Richard Courant. However, because Faber had announced his proof already, it was not deemed sufficient for the degree to be awarded. Krahn subsequently provided a proof of the conjecture in higher dimensions in 1926, for which he got his doctorate. To make sure his result is published fast, Krahn submitted it to *Acta Comm. Univ. Tartu (Dorpat)* [26], a journal run by the University of Tartu in Estonia where Krahn had studied.

The main obstacle for the proof in higher dimensions was that the isoperimetric inequality between surface area and volume was not available at the time. Hence, the bulk of Krahn's paper [26] is devoted to a proof of the isoperimetric inequality using induction by the dimension. The proof of Rayleigh's conjecture does not occupy much space. Usually the proof of the isoperimetric inequality in higher dimensions is attributed to Erhard Schmidt [38]. The Review of that paper in Zentralblatt (Zbl 0020.37301) says "*Die Arbeit enthält den ersten strengen Beweis für die isoperimetrische Ungleichheit in einem Euklidischen Raum von mehr als drei Dimensionen.*"[1] (see also the comments in [28, p. 89]). We refer to [28] for an English translation of Krahn's work and more on his biography. In Chapter 6 that book also contains a discussion of Krahn's proof, but with a rather different emphasis.

Both proofs, the one by Faber and the one by Krahn employ a similar idea. However, Faber implements it by discretising the integrals involved and then passes to the limit, whereas Krahn uses ideas now part of geometric measure theory, in particular the co-area formula. Krahn's techniques are now

[1] "The paper contains the first rigorous proof of the isoperimetric inequality in Euclidean space of more than three dimensions."

widely used and have been rediscovered again. For instance, one key argument in the celebrated paper [40] by Talenti on the best constants of Sobolev inequalities recovers *exactly* a generalised version of Krahn's approach. Finally, Krahn's argument provides the uniqueness of the minimising domain, whereas Faber's approximation argument does not.

The result is now referred to as the *Rayleigh-Faber-Krahn inequality* or simply the *Faber-Krahn inequality* for the first eigenvalue $\lambda_1(\Omega)$ of

$$-\Delta u = \lambda u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain. It is well known that $\lambda_1(\Omega)$ is simple and the eigenfunction ψ can be chosen positive. The result as we know it today can be stated as follows.

Theorem 1.1 (Rayleigh-Faber-Krahn inequality). *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain and Ω^* an open ball of the same volume. Then $\lambda_1(\Omega) \geq \lambda_1(\Omega^*)$ with equality if and only if Ω is a ball except possibly for a set of capacity zero.*

Krahn assumes that Ω has a piecewise analytic boundary, but this is not necessary for his proof to work. The uniqueness of the minimising domain is discussed in Remark 5.1. In [26, Section 4] Krahn also proves that the union of two disjoint discs of the same area minimises the second eigenvalue, a fact sometimes attributed to Szegö as remarked in [21, Section 4.1].

The above theorem serves as a prototype for a huge range of related problems more generally called *isoperimetric inequalities*. It also serves as a prototype for a *shape optimisation problem* such as those in [7]. Despite the large number of known results there are still many challenging conjectures in the area too numerous to list here. For references and more on the history of the problem the survey articles by Osserman [30] and Payne [31] are still a good source. A classic is also Bandle's book [4]. More recent surveys include [3, 21]. The old classic is the book [33] by Pólya and Szegö. Similar problems are also studied on manifolds, see for instance the book by Chavel [10].

The purpose of these notes is to look at Krahn's original proof, translate it into more modern terminology and see how the idea works under weaker assumptions. In particular we see that Krahn is using an identity equivalent to what is now known as the co-area formula. We also fill in the missing technical detail. The basic steps are as follows:

1. Introduce a coordinate transformation which turns out to be equivalent to the co-area formula from geometric measure theory.
2. Apply this coordinate transformation to rewrite $\|\psi\|_2^2$, $\|\nabla\psi\|_2^2$ and $F(t) := \int_{U_t} 1\,dx$, where $U_t = \{x \in \Omega \colon \psi > t\}$.
3. Using the isoperimetric inequality derive a lower bound for the first eigenvalue of (1.1) in terms of $F(t)$.
4. Rearrange the eigenfunction to get a radially symmetric function, and observe that $F(t)$ is the same as the original function. Use the previous step and the fact that the minimum of the Rayleigh quotient is the first eigenvalue.
5. Argue that $\lambda_1(\Omega) = \lambda_1(\Omega^*)$ implies that Ω is a ball.

We discuss and comment on each step separately, and then in a final section we show how to generalise the arguments for bounded domains in $\mathbb{R}^N$. Note that Steps 1–4 do not depend on looking at an eigenvalue problem. The results apply to Schwarz symmetrisation of functions in general and can be used for other purposes like finding the best constant in Sobolev inequalities as in [40] or for nonlinear problems such as [14].

2. Heuristic derivation of the co-area formula

In this section we discuss a coordinate transformation used in the first part of Krahn's paper [25, pages 98/99], and which appears in a more disguised form in [26] in higher dimensions. We show that this transformation is essentially the co-area formula. Krahn applies it to the first eigenfunction $\psi > 0$ of (1.1) and starts with the observation that it has at most finitely many critical points, that is, points where $\nabla\psi = 0$. Hence there are at most finitely many levels $t_1, \dots, t_n$ such that

$$S_t := \{x \in \Omega \colon \psi(x) = t\} \tag{2.1}$$

is not a smooth level curve in Ω (Krahn uses z for the levels). Consider now a point $(x, y) \in \Omega$ so that $\nabla\psi(x, y) \neq 0$ and set $t = \psi(x, y)$. The implicit function theorem implies that a smooth level curve passes through that point and all points nearby. Hence a neighbourhood of that point can be parametrised by $\Phi(s, t)$, where s is arc length on the level curve and t is the level in the direction of the gradient orthogonal to the level curve S_t. Locally the coordinate transformation is as shown in Figure 2. If we use the

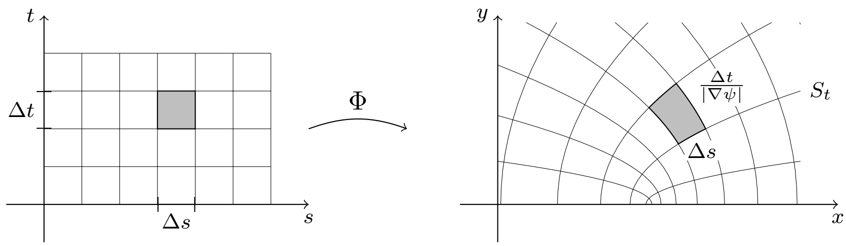

FIGURE 2. Local coordinate transformation flattening the level curves

transformation formula for integrals to write an integral over Ω as integral with respect to the coordinates (s, t), then we need the determinant of the Jacobian matrix J_Φ of Φ

$$D := |\det J_\Phi(s, t)|$$

as defined in Krahn's paper. The Jacobian determinant of Φ^{-1} is

$$|\det J_{\Phi^{-1}}(s, t)| = |\det J_\Phi^{-1}(s, t)| = \frac{1}{D} = |\nabla\psi| \tag{2.2}$$

Graphically, in Figure 2, we can see why: The line element Δs remains unchanged whereas in the direction of the gradient (perpendicular to S_t) the line element Δt is stretched by the factor $1/|\nabla\psi|$. The reason is that $|\nabla\psi|$ is the slope of the graph of ψ in the direction of $\nabla\psi$. Hence the rectangle of side lengths Δs and Δt is transformed into a small rectangle of side lengths Δs and $\frac{\Delta t}{|\nabla\psi|}$. For a given measurable function $u \in L_1(\Omega)$ we can then write

$$\int_\Omega u\,dx = \int_{\mathbb{R}}\int_{S_t} uD\,ds\,dt. \tag{2.3}$$

Krahn applies this formula in the special cases $u \equiv 1$, $u = |\psi|^2$ and $u = |\nabla\psi|^2$. He also writes the line integral by assuming S_t is parametrised by arc length as an integral from zero to $L(z)$, the length of the curve S_t for $t = z$. In our exposition we write the line integral without a particular parametrisation and replace $1/D$ by $|\nabla\psi|$. If we rewrite (2.3) using (2.2), then for any measurable function $g\colon \Omega \to [0,\infty)$

$$\int_\Omega g|\nabla\psi|\,dx = \int_0^\infty \int_{S_t} g\,ds\,dt. \tag{2.4}$$

The reasoning above is also valid in higher dimensions. Then S_t is a hypersurface and ds is replaced by the surface measure $d\sigma$ on S_t. Moreover, we can admit sign changing measurable functions g if $g|\nabla\psi|$ is integrable. However, it seems not completely clear that this local transformation can be made global by subdividing the domain and dealing with the exceptional levels where $\nabla\psi$ vanishes, unless ψ is very smooth.

As pointed out in [31, page 453] it was unclear at the time whether the above transformation can be properly justified. The formula (2.4) however is a special case of the *co-area formula* due to Federer [18, Theorem 3.1]. It is valid if ψ is Lipschitz continuous on Ω, and it is not required that ψ be zero on $\partial\Omega$. In that case, the usual surface measure has to be replaced by the $(N-1)$-dimensional Hausdorff measure. Standard references for the co-area formula include Federer's book on geometric measure theory [19] and the book [16] by Evans and Gariepy. The latter is more accessible. In [29, Section 1.2.4] Maz'ya gives a proof of (2.4) for $\psi \in C^\infty(\Omega)$ based on Sard's lemma and the divergence theorem. More general versions of the co-area formula for functions of bounded variation appear for instance in [1, 20].

3. Consequences of the co-area formula

Let $\psi\colon \Omega \to [0,\infty)$ be Lipschitz continuous and S_t the level surface as defined in (2.1). If $g\colon \Omega \to [0,\infty)$ is measurable, then the co-area formula implies

$$\int_\Omega \frac{g}{|\nabla\psi| + \varepsilon}|\nabla\psi|\,dx = \int_0^\infty \int_{S_t} \frac{g}{|\nabla\psi| + \varepsilon}\,d\sigma\,dt$$

for all $\varepsilon > 0$. Letting $\varepsilon \to 0$, by the monotone convergence theorem

$$\int_\Omega g\,dx = \int_0^\infty \int_{S_t} \frac{g}{|\nabla\psi|}\,d\sigma\,dt. \tag{3.1}$$

As Krahn did, we can apply the above formula to the level sets

$$U_t := \{x \in \Omega \colon \psi(x) > t\}$$

and the constant function $g \equiv 1$. We then get

$$F(t) := \int_{U_t} 1\,dx = \int_t^\infty \int_{S_\tau} \frac{1}{|\nabla\psi|}\,d\sigma\,d\tau = \int_t^\infty \int_{S_\tau} D\,d\sigma\,d\tau. \tag{3.2}$$

Since Ω has bounded measure the above shows that the function

$$t \to \int_{S_t} \frac{1}{|\nabla\psi|}\,d\sigma$$

is integrable on $(0,\infty)$ and therefore F is absolutely continuous. Hence F is differentiable almost everywhere and

$$F'(t) = -\int_{S_t} \frac{1}{|\nabla\psi|}\,d\sigma > 0 \tag{3.3}$$

for almost all $t \in [0, \|\psi\|_\infty)$ (see [37, Theorem 8.17]). The strict inequality comes since $F(t) < \infty$ implies that $|\nabla\psi| > 0$ on S_t for almost all $t \in (0, \|\psi\|_\infty)$ because otherwise the integral on the right hand side of (3.2) would not be finite. Krahn does not argue rigorously here, but just differentiates F to get the above expression for $F'(t)$. From the above and (3.1), Krahn gets a formula for $\|\psi\|_2^2$ and $\|\nabla\psi\|_2^2$, namely

$$\|\psi\|_2^2 = \int_\Omega |\psi|^2\,dx = \int_0^\infty \tau^2 \int_{S_\tau} \frac{1}{|\nabla\psi|}\,d\sigma\,d\tau = \int_0^\infty \tau^2 F'(\tau)\,d\tau \tag{3.4}$$

and

$$\|\nabla\psi\|_2^2 = \int_\Omega |\nabla\psi|^2\,dx = \int_0^\infty \int_{S_\tau} |\nabla\psi|\,d\sigma\,d\tau. \tag{3.5}$$

4. A lower estimate for the Dirichlet integral

The next step in Krahn's proof is a lower estimate for the Dirichlet integral $\|\nabla\psi\|_2^2$. We again assume that ψ is a non-negative Lipschitz function on $\Omega \subset \mathbb{R}^2$ because the argument Krahn gives does not rely on analyticity or the fact that ψ is the first eigenfunction of (1.1). Krahn argues with the arithmetic-geometric mean inequality $4a^2 \leq (\xi + a^2/\xi)^2$ for all $a \geq 0$ and $\xi > 0$. However, it seems easier to argue with Young's inequality

$$a \leq \frac{1}{2\xi} + \frac{a^2\xi}{2}$$

valid for all $a \geq 0$ and $\xi > 0$. Note that there is equality if and only if $a = \xi$. Applying this for $a = -F'(t)/\sigma(S_t)$ and $\xi = |\nabla\psi|$ and integrating over S_t we get

$$\begin{aligned}
-2F'(t) &\leq \int_{S_t} \frac{1}{|\nabla\psi|}\,d\sigma + \Big(\frac{F'(t)}{\sigma(S_t)}\Big)^2 \int_{S_t} |\nabla\psi|\,d\sigma \\
&= -F'(t) + \Big(\frac{F'(t)}{\sigma(S_t)}\Big)^2 \int_{S_t} |\nabla\psi|\,d\sigma,
\end{aligned}$$

where we used (3.3) for the last equality. We have used that $\nabla\psi \neq 0$ on S_t for almost all $t > 0$ as proved in the previous section. Rearranging we get the key inequality

$$-\frac{\bigl[\sigma(S_t)\bigr]^2}{F'(t)} \leq \int_{S_t} |\nabla\psi|\, d\sigma \tag{4.1}$$

for almost all $t > 0$ with equality if and only if $-F'(t)/\sigma(S_t) = |\nabla\psi|$ almost everywhere on S_t. In particular we have equality if ψ is radially symmetric, and equality implies that $|\nabla\psi|$ is constant on S_t. We then apply the isoperimetric inequality

$$\bigl[\sigma(S_t)\bigr]^2 \geq 4\pi F(t) \tag{4.2}$$

for almost all t and therefore

$$-4\pi\frac{F(t)}{F'(t)} \leq \int_{S_t} |\nabla\psi|\, d\sigma. \tag{4.3}$$

If we integrate the above inequality we get, using (3.5),

$$-4\pi\int_0^\infty \frac{F(t)}{F'(t)}\, dt \leq \int_0^\infty \int_{S_t} |\nabla\psi|\, d\sigma\, dt = \|\nabla\psi\|_2^2. \tag{4.4}$$

Applied to the first eigenfunction of (1.1) we get the following estimate on the first eigenvalue.

Proposition 4.1. *Let $\lambda_1(\Omega)$ denote the first eigenvalue of (1.1) and ψ a positive eigenfunction normalised so that $\|\psi\|_2^2 = 1$. Then*

$$\lambda_1(\Omega) = \|\nabla\psi\|_2^2 \geq -4\pi\int_0^\infty \frac{F(t)}{F'(t)}\, dt \tag{4.5}$$

with equality if and only if U_t is a disc and $|\nabla\psi|$ is constant on S_t for almost all $t > 0$.

5. The symmetrisation argument

We now assume that ψ is the first eigenfunction of (1.1) normalised so that $\psi > 0$ and $\|\psi\|_2 = 1$. Krahn constructs a new function ψ^* replacing the sets U_t by a disc U_t^* of the same area. This is what we now call the *Schwarz symmetrisation* of ψ. The precise definition is

$$\psi^*(x) := \sup\{t \in \mathbb{R}\colon |U_t| > \pi|x|^2\},$$

but Krahn is much less formal in his paper. Since $\bigcup_{s>t} U_s = U_t$ the function $t \mapsto F(t)$ is right continuous. From that it is not hard to deduce that $F(t) = |U_t^*|$ if we set

$$U_t^* = \{x \in \Omega^*\colon \psi^*(x) > t\}.$$

Hence the right hand side of (4.5) is the same for ψ and ψ^*. However, since ψ^* is radially symmetric Krahn gets

$$\lambda_1(\Omega) \geq -4\pi\int_0^\infty \frac{F(t)}{F'(t)}\, dt = \|\nabla\psi^*\|_2^2 \geq \lambda_1(\Omega^*),$$

where the last inequality follows since $\lambda_1(\Omega)$ is the minimum of the Rayleigh quotient and $\|\psi^*\|_2 = \|\psi\|_2 = 1$ by (3.4). Hence

$$\lambda_1(\Omega) \geq \lambda_1(\Omega^*)$$

as claimed. If there is equality, then from the above

$$\lambda_1(\Omega) = -4\pi \int_0^\infty \frac{F(t)}{F'(t)}\,dt = \|\nabla\psi\|_2^2$$

and so Proposition 4.1 implies that almost all level sets U_t are nested discs if Ω has continuous boundary. Hence $\Omega = \bigcup_{t>0} U_t$ is a disc as well.

Remark 5.1. The above argument shows that $\|\nabla\psi\|_2 = \|\nabla\psi^*\|_2$. There are theorems which ensure that this implies that $\psi = \psi^*$ almost everywhere (see [6]). We do not make use of these. We show that Ω is the union of nested balls and therefore a ball. It is irrelevant whether or not ψ is radially symmetric. However, the radial symmetry of ψ follows from the simplicity of the first eigenvalue once we know Ω is a ball.

The uniqueness of the minimising domain is not stated in many standard references because sources like [33] obtain the inequality $\|\nabla\psi^*\|_2 \leq \|\nabla\psi\|_2$ in a different way. If it is stated, then often not with specific regularity assumptions on Ω. We refer to [22, pages 92/93] for a discussion.

If Ω is an arbitrary bounded open set, then the equality of the eigenvalues implies that U_t and therefore Ω is a ball up to a set of $(N-1)$-dimensional Hausdorff measure zero. Equality up to a set of capacity zero then follows from the main result in [2]. An alternative proof of uniqueness appears in [15], including a characterisation of all symmetric operators in divergence form minimising the first eigenvalue.

We have seen that at the time it was not clear whether or not the co-area formula can be justified. Even if it could be justified for analytic functions, it was not clear whether or not this applies to ψ^* as well. To avoid these issues Tonelli gave a slightly different proof under weaker assumptions on the domain and the function by showing that the surface area of the graph of ψ is larger than that of the graph of ψ^*. He then also obtained (4.4) (see [41, page 261]). A similar approach is taken by Pólya [32, 34]. The knowledge and techniques up to 1950 is collected in Pólya and Szegö's book [33].

For Krahn's argument to work we only need that ψ^* is Lipschitz continuous because then the co-area applies. The Lipschitz continuity of ψ^* follows relatively easily from the Brunn-Minkowski inequality as for instance shown in [39, Lemma 1] or [4, Lemma 2.1] or [27] (for one dimension see also [22, Lemma 2.3]). For general properties of Schwarz symmetrisation see for instance [4, 22, 24, 27, 40].

6. Modification for higher dimension

As mentioned in Section 1.1, Krahn proves Theorem 1.1 also in dimensions $N \geq 3$ in his PhD thesis published in [26]. The main part of Krahn's paper

[26, Section 2] is to establish the isoperimetric inequality

$$\sigma(\partial U) \leq C(N)|U|^{1-1/N} \tag{6.1}$$

for sets $U \subseteq \mathbb{R}^N$ with piecewise analytic boundary, where $|U|$ is the measure of U and $\sigma(\partial U)$ the surface area of ∂U. The isoperimetric constant $c(N)$ is such that the above is an equality if and only if U is a ball. In [26, Section 3] Krahn then proves (4.1), again by an approach equivalent to using the co-area formula. In his writing, for a the problem in n dimensions,

$$\sum_{i=1}^{k} \int_{B_i(u)} \frac{M_i^2}{D_i} dv_{i1} \dots dv_{in-1} \geq -\frac{b(u)^2}{Q'(u)}$$

with equality if and only if U_t is a ball. (Note that M_i^2/D_i^2 should be M_i^2/D_i in the integrals in [28, pp 170–172]). In our notation $u = t$, $b(u) = \sigma(S_t)$ and $Q(u) = F(t) = |U_t|$. Moreover, S_t is written as a disjoint union of graphs of analytic functions parametrised by v_{ij}, $j = 1, \dots n-1$, over $B_i(u)$. As shown earlier in Krahn's paper, $M_i/D_i = |\nabla\psi|$, and $M_i dv_1 \dots dv_{n-1}$ is the surface measure $d\sigma$ on S_t. Hence the above inequality corresponds exactly to (4.1). Applying (6.1) we then get

$$\int_{S_t} |\nabla\psi|\, d\sigma \geq -\frac{\big[\sigma(S_t)\big]^2}{F'(t)} \geq -c(N)^2 \frac{F(t)^{2(1-1/N)}}{F'(t)}$$

for almost all $t > 0$. Integrating, we get an estimate similar to (4.5), that is,

$$\|\nabla\psi\|_2^2 \geq -\int_0^\infty \frac{\big[\sigma(S_t)\big]^2}{F'(t)}\, dt \geq -c(N)^2 \int_0^\infty \frac{F(t)^{2(1-1/N)}}{F'(t)}\, dt \tag{6.2}$$

with equality in the second estimate if and only if S_t is a sphere for almost all $t > 0$. Hence, if ψ^* is the symmetrisation of ψ, then similarly as in Krahn's proof in two dimensions

$$\lambda_1(\Omega) = \|\nabla\psi\|_2^2 \geq -c(N)^2 \int_0^\infty \frac{F(t)^{2(1-1/N)}}{F'(t)}\, dt = \|\nabla\psi^*\|_2^2 \geq \lambda_1(\Omega^*).$$

The uniqueness of the minimising domain follows as in Section 5. We also note that Krahn's approach essentially coincides with Talenti's key argument in finding the best constant in the Sobolev inequality, see [40, pages 361/362]. The inequality (21) in Talenti's paper is a generalisation of (6.2) for arbitrary $p \in (1, \infty)$. However, in his paper there is no reference to Krahn's work, which suggests that Talenti was not aware of Krahn's approach. Note also that Talenti's generalisation gives a proof of Theorem 1.1 for the first eigenvalue of the p-Laplace operator for $p \in (1, \infty)$.

7. Concluding Remarks

The basic idea of Krahn's proof is to get an estimate of the form $\lambda_1(\Omega) \geq L(\psi)$, where $L(\psi)$ is a function of ψ only depending on the volume of the level sets U_t with equality if ψ is radially symmetric. The way to achieve this is to use the co-area formula to link to the geometry of the domain via properties of the level sets U_t.

That idea also works for other problems, as for instance the analogous problem for the elastically supported membrane, a conjecturex that seems to go back to Krahn as well. The boundary conditions are

$$\frac{\partial \psi}{\partial \nu} + \beta \psi = 0$$

on $\partial\Omega$, where ν is the outer unit normal and $\beta \in (0, \infty)$ constant. For Dirichlet boundary conditions we wrote the eigenvalue in terms of the Dirichlet integral. This does not seem to work in case of Robin boundary conditions (the elastically supported membrane). The reason is that the eigenfunction is not constant on $\partial\Omega$ and therefore S_t is not in general the boundary of U_t. Hence (6.2) does not apply.

The remedy is to write $\lambda_1(\Omega)$ in terms of a functional involving the whole boundary of U_t, not just the part inside Ω. More precisely,

$$\lambda_1(\Omega) = \frac{1}{F(t)} \Big(\int_{\partial U_t \cap \Omega} \frac{|\nabla \psi|}{\psi} \, d\sigma + \int_{\partial U_t \cap \partial\Omega} \beta \, d\sigma - \int_{U_t} \frac{|\nabla \psi|^2}{\psi^2} \, dx \Big)$$

for almost all $t > 0$. Then a similar idea can be applied. The first successful attempt is the proof by Bossel [5] in two dimensions using ideas from extremal length in complex analysis, but there were still a lot of technical issues to be resolved. Higher dimensions, more general domains and the p-Laplace operators are treated in [8, 9, 12, 13, 15]. The common feature with Krahn's approach is that the co-area formula provides the essential link to the isoperimetric inequality.

Finally we note that for the above boundary conditions the minimising domain for the second eigenvalue is the union of equal balls as shown in [23]. The idea of the proof is similar to that used for Dirichlet boundary conditions.

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