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Non-positivity of the semigroup generated by the Dirichlet-to-Neumann operator

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Abstract By analysing some explicit examples we investigate the positivity and the non-positivity of the semigroup generated by the Dirichlet-to-Neumann operator associated with the operator $\Delta + \lambda I$ as λ varies. It is known that the semigroup is positive if $\lambda < \lambda_1$, where λ_1 is the principal eigenvalue of $-\Delta$ with Dirichlet boundary conditions. We show that it is possible for the semigroup to be non-positive, eventually positive or positive and irreducible depending on $\lambda > \lambda_1$.

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Keywords Dirichlet-to-Neumann operator · positive semigroup · eventually positive semigroup · principal eigenvalue

1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded open set with smooth boundary, and let $\lambda \in \mathbb{R}$. The Dirichlet-to-Neumann operator D_λ is a closed operator on $L^2(\partial\Omega)$ defined as follows. Given $\varphi \in H^{1/2}(\Omega)$ solve the Dirichlet problem

$$\begin{aligned} \Delta u + \lambda u &= 0 \quad \text{in } \Omega, \\ u &= \varphi \quad \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

A solution only exists if λ is not an eigenvalue of $-\Delta$ with Dirichlet boundary conditions. If u is smooth enough we define

$$D_\lambda \varphi := \frac{\partial u}{\partial \nu}, \tag{1.2}$$

where ν is the outer unit normal to $\partial\Omega$. One can show D_λ extends uniquely to an operator $D_\lambda \in \mathscr{L}\big(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)\big)$. If we denote its part in $L^2(\partial\Omega)$ again by

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D_λ, then $-D_\lambda$ generates an analytic semigroup e^{-tD_λ} on $L^2(\partial\Omega)$ and also on $C(\partial\Omega)$; see for instance [4,5,10,12].

Let $0<\lambda_1<\lambda_2<\lambda_3<\dots$ be the strictly ordered Dirichlet eigenvalues of $-\Delta$ on Ω. It is shown in [4,5] that e^{-tD_λ} is positive and irreducible if $\lambda<\lambda_1$. The question left open was whether or not the semigroup is positive for any $\lambda>\lambda_1$. It is tempting to conjecture that it is not positive for $\lambda>\lambda_1$. The aim of this paper is to explore this question by means of two explicit examples, and to show that there is no obvious conjecture. The examples should stimulate more research into positivity properties of e^{-tD_λ} on general domains.

The first example is a bounded interval $\Omega=(0,L)$, where $L>0$. In that case positivity and non-positivity alternate at each Dirichlet eigenvalue $\lambda_k=(k\pi/L)^2$. Details are given in Section 3. One could therefore conjecture that positivity and non-positivity alternate at each eigenvalue, possibly counting multiplicity. One could also conjecture that if e^{-tD_λ} is a positive semigroup for some $\lambda\in(\lambda_k,\lambda_{k+1})$, then e^{-tD_λ} is a positive semigroup on the whole interval $(\lambda_k,\lambda_{k+1})$. Such conjectures are disproved by the second example.

Theorem 1.1 *Let $\Omega=B(0,1)$ be the unit disc in $\mathbb{R}^2$. Then the semigroup e^{-tD_λ} has the following properties.*

(i) *The semigroup e^{-tD_λ} is positive and irreducible for all $\lambda<\lambda_1$, and for λ in a left neighbourhood of every simple eigenvalue.*
(ii) *The semigroup e^{-tD_λ} is eventually positive and irreducible for all $\lambda\in(\lambda_3,\lambda_4)$. More precisely, there exists $T>0$ such that e^{-tD_λ} is positive and irreducible for all $t\geq T$ and all $\lambda\in(\lambda_3,\lambda_4)$.*
(iii) *The semigroup e^{-tD_λ} is not positive for λ in a neighbourhood of every double eigenvalue, and in a right neighbourhood of every simple eigenvalue.*

Note that all Dirichlet eigenvalues on the disc are of multiplicity one or two; see Section 4. From Theorem 1.1 we deduce the following facts.

1. The semigroup e^{-tD_λ} can change from not positive to positive between two eigenvalues. According to the above theorem this is the case for $\lambda\in(\lambda_3,\lambda_4)$.
2. It is possible that e^{-tD_λ} is positive (and irreducible) for large enough t, but not for small t.
3. In order for e^{-tD_λ} to be positive (or even eventually positive), the smallest eigenvalue $\mu_1(\lambda)$ of D_λ must have a positive eigenfunction. We see in Section 4 why this is only possible in a left neighbourhood of every simple eigenvalue.
4. It may be that the semigroup e^{-tD_λ} is eventually positive for all $\lambda\in(\lambda_{k-1},\lambda_k)$ if λ_k is a simple eigenvalue. This is however not entirely clear; see Remark 4.7.

Proofs and further discussions of the example of the disc are given in Section 4. The analysis of the explicit examples gives insight into what could be true in general. However, one has to be careful not to infer too much from the multiplicities of the eigenvalues. Due to the symmetry of the ball we expect higher multiplicity, but for a generic domain, the eigenvalues are all simple; see [21,13]. It just happens that on the ball the simple eigenvalues have the properties required for the positivity of the semigroup. Supporting evidence for the comments below is given in Section 5.

1. An important necessary condition for the positivity of the semigroup is that the smallest eigenvalue of D_λ has a positive eigenfunction. The condition is not sufficient as the example of the disc shows.
2. There is a good chance that a positive or eventually positive semigroup is (eventually) irreducible. Standard results on positive irreducible operators (Krein-Rutman Theorems) then imply the simplicity and uniqueness of the eigenvalue with positive eigenfunction. Hence we only expect positivity near simple eigenvalues.
3. We do not expect positivity of the semigroup near every simple eigenvalue because the corresponding eigenfunctions have changing sign most of the time. Note that every eigenvalue of D_λ corresponds to an eigenvalue of $-\Delta$ with Robin boundary conditions, that is, if $D_\lambda \varphi = \mu\varphi$, and u solves (1.1), then
$$\begin{aligned} -\Delta u &= \lambda u \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} - \mu u &= 0 \quad \text{on } \partial\Omega; \end{aligned} \tag{1.3}$$
see [5]. Hence, for the minimal eigenvalue of D_λ to have a positive eigenfunction, a corresponding eigenfunction u of the Robin problem needs to have a positive (or negative) trace. This is not a generic property even if the eigenvalue is simple.
4. For most domains, the eigenfunctions to the second eigenvalue of the Dirichlet or Robin problem do not have a closed nodal line. Hence the eigenfunction of D_λ which is the trace of that eigenfunction has changing sign. In such a situation the semigroup e^{-tD_λ} is not positive for $\lambda \in (\lambda_1, \lambda_2)$. There are however examples where the nodal line is closed and therefore the trace is positive; see [16]. Nevertheless, even in that case we do not expect positivity or even eventual positivity of e^{-tD_λ} for λ in a right neighbourhood of λ_1. However, we do expect eventual positivity or even positivity for some $\lambda \in (\lambda_1, \lambda_2)$.

We finally note that our results provide counter-examples to a claim in [12], where the main theorem implies that e^{-tD_λ} is a positive semigroup regardless of the value of λ. The proof in [12] is correct for $\lambda < \lambda_1$, but does not work for $\lambda > \lambda_1$ due to an incorrect application of the maximum principle in the appendix of [11]. Positivity in the case $\lambda = 0$ is also proved in [10].

2 The Dirichlet-to-Neumann operator

We briefly outline how the Dirichlet-to-Neumann operator D_λ can be defined by means of a bilinear form. Given φ and ψ, solve (1.1) for boundary values φ and ψ. This gives functions $u, v \in H^1(\Omega)$. We then let
$$a_\lambda(\varphi, \psi) := \int_\Omega \nabla u \nabla \bar{v} - \lambda u \bar{v}\, dx. \tag{2.1}$$
This form turns out to be bounded on $H^{1/2}(\partial\Omega)$. If the domain and the functions involved are smooth enough, then we can apply the divergence theorem and (1.1) to conclude that
$$a_\lambda(\varphi, \psi) = -\int_\Omega (\Delta u + \lambda u)\bar{v}\, dx + \int_{\partial\Omega} \bar{\psi} \frac{\partial u}{\partial \nu}\, d\sigma = \int_{\partial\Omega} \bar{\psi} D_\lambda \varphi\, d\sigma. \tag{2.2}$$

Hence D_λ is the operator induced by the form a_λ on $L^2(\partial\Omega)$. A rigorous treatment of this approach on non-smooth domains can be found in [2,4,5]. To analyse the positivity of e^{-tD_λ} we either compute the semigroup explicitly, or we use the Beurling-Deny criterion. That criterion asserts that $e^{-tD_\lambda} \geq 0$ if and only if

$$a_\lambda(\varphi^+, \varphi^-) \leq 0$$

for all $\varphi \in D(a_\lambda)$; see [18, Theorem 2.6]. Hence if we can find φ such that $a_\lambda(\varphi^+, \varphi^-) > 0$, then the semigroup cannot be positive. We use this criterion to prove non-positivity in a left or right neighbourhood of each eigenvalue of the Dirichlet Laplacian on the disc.

From the form we can also say something about the stability of the semigroup e^{-tD_λ}; see also [4].

Proposition 2.1 *The semigroup e^{-tD_λ} is exponentially stable if $\lambda < 0$, and unstable if $\lambda > 0$.*

Proof The stability is determined by the sign of the first eigenvalue of D_λ which is given by the infimum of the Rayleigh coefficient

$$\mu_1(\lambda) = \inf_{\varphi \in H^{1/2}(\partial\Omega)} \frac{a_\lambda(\varphi, \varphi)}{\|\varphi\|_2^2}.$$

Fix $\varphi \in H^{1/2}(\partial\Omega)$ and let $u \in H^1(\Omega)$ be the solution of (1.1). If $\lambda \leq 0$, then

$$a_\lambda(\varphi, \varphi) = \int_\Omega |\nabla u|^2 - \lambda |u|^2 \, dx \geq \int_\Omega |\nabla u|^2 \, dx \geq 0,$$

so that $\mu_1(\lambda) \geq 0$. If $\lambda = 0$ and $\varphi = c$ is constant, then $u = c$ and $\mu(\lambda) = 0$. Due to the compactness of the embedding $H^{1/2}(\partial\Omega) \hookrightarrow L^2(\partial\Omega)$ we have $\mu_1(\lambda) > 0$ if $\lambda < 0$, and $\mu_1(\lambda) < 0$ if $\lambda > 0$. This implies the exponential stability of e^{-tD_λ} for $\lambda < 0$ and the instability for $\lambda > 0$. $\square$

It is interesting to note that the change of stability does not occur at an eigenvalue of the Dirichlet Laplacian. The change of stability occurs at the first (the trivial) eigenvalue of the Neumann Laplacian. The Neumann and the Robin eigenvalues $-\Delta$ influence the precise behaviour of e^{-tD_λ}. There is a close connection between the spectrum of D_λ and the spectrum of the Robin Laplacian as shown in [5]. Our examples provide some insight into its effect on the positivity properties and the precise behaviour of the semigroup e^{-tD_λ}, or the operator e^{-tD_λ} for some range of t.

3 Non-positivity in one dimension

We explicitly compute the Dirichlet-to-Neumann operator on an interval $(0,L)$, $L > 0$ and show that the positivity and the non-positivity of e^{-tD_λ} change at every eigenvalue.

If $\Omega = (0,L)$, then the boundary value problem (1.1) reduces to

$$u'' + \lambda u = 0 \quad \text{on } (0,L), \qquad u(0) = a, \; u(L) = b. \tag{3.1}$$

The boundary $\partial\Omega = \{0, L\}$ consists of two points and therefore we can identify $L^2(\partial\Omega)$ with $\mathbb{R}^2$ in a natural way. We have to find solutions u_1, u_2 of (3.1) corresponding to the standard basis of $\mathbb{R}^2$, that is, $u_1(0) = u_2(L) = 1$ and $u_1(L) = u_2(0) = 0$. If $\lambda > 0$ these solutions obviously are

$$u_1(x) := \frac{\sin\sqrt{\lambda}(L-x)}{\sin\sqrt{\lambda}L} \quad \text{and} \quad u_2(x) := \frac{\sin\sqrt{\lambda}x}{\sin\sqrt{\lambda}L}.$$

If $\lambda < 0$ we get

$$u_1(x) := \frac{\sinh\sqrt{-\lambda}(L-x)}{\sinh\sqrt{-\lambda}L} \quad \text{and} \quad u_2(x) := \frac{\sinh\sqrt{-\lambda}x}{\sinh\sqrt{-\lambda}L},$$

and if $\lambda = 0$, we get $u_1(x) = (x-L)/L$ and $u_2(x) = x/L$. A unique solution of (3.1) only exists if $\sqrt{\lambda}L \neq k\pi$ for all $k \in \mathbb{N}$, that is,

$$\lambda \neq \lambda_k := \left(\frac{k\pi}{L}\right)^2.$$

Hence if $\lambda \neq \lambda_k$, then the solution of (1.1) is $u(x) = au_1(x) + bu_2(x)$. To determine the matrix representation of D_λ we compute the coordinate vectors of the images of the basis vectors. If $\lambda > 0$, $\lambda \neq \lambda_k$, they are

$$D_\lambda \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -u_1'(0) \\ u_1'(L) \end{bmatrix} = \frac{\sqrt{\lambda}}{\sin\sqrt{\lambda}L} \begin{bmatrix} \cos\sqrt{\lambda}L \\ -1 \end{bmatrix}$$

and

$$D_\lambda \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -u_2'(0) \\ u_2'(L) \end{bmatrix} = \frac{\sqrt{\lambda}}{\sin\sqrt{\lambda}L} \begin{bmatrix} -1 \\ \cos\sqrt{\lambda}L \end{bmatrix}.$$

The minus sign in the first component comes from the fact that the outer unit normal is pointing in the negative direction at $x = 0$, and in the positive direction at $x = L$. Hence, the matrix representation of D_λ is

$$D_\lambda = \frac{\sqrt{\lambda}}{\sin\sqrt{\lambda}L} \begin{bmatrix} \cos\sqrt{\lambda}L & -1 \\ -1 & \cos\sqrt{\lambda}L \end{bmatrix}.$$

As expected, the matrix is symmetric because D_λ is a self-adjoint operator. Even more, the matrix is of the form

$$\begin{bmatrix} \alpha & -\beta \\ -\beta & \alpha \end{bmatrix} = \alpha \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & \beta \\ \beta & 0 \end{bmatrix}.$$

with

$$\alpha(\lambda) := \frac{\sqrt{\lambda}\cos\sqrt{\lambda}L}{\sin\sqrt{\lambda}L} \quad \text{and} \quad \beta(\lambda) := \frac{\sqrt{\lambda}}{\sin\sqrt{\lambda}L} \tag{3.2}$$

if $\lambda > 0$. For $\lambda < 0$ we get in a similar manner

$$\alpha(\lambda) := \frac{\sqrt{-\lambda}\cosh\sqrt{-\lambda}L}{\sinh\sqrt{-\lambda}L} \quad \text{and} \quad \beta(\lambda) := \frac{\sqrt{-\lambda}}{\sinh\sqrt{-\lambda}L}. \tag{3.3}$$

Note that (3.3) is the same as (3.2) with complex argument $\sqrt{\lambda} = i\sqrt{-\lambda}$. Hence

$$e^{-tD_\lambda} = e^{-t\alpha} \begin{bmatrix} \cosh\beta t & \sinh\beta t \\ \sinh\beta t & \cosh\beta t \end{bmatrix}. \tag{3.4}$$

That semigroup is positive if and only if $\beta \geq 0$. Therefore, e^{-tD_λ} is positive if and only if $\lambda \leq 0$ or if $\lambda_{2k} < \lambda < \lambda_{2k+1}$ for some $k \in \mathbb{N}$. This means positivity and non-positivity change at each Dirichlet eigenvalue λ_k of $-\Delta$. For the stability of the semigroup we need to compute the eigenvalues of D_λ. The eigenvalues of that matrix are $\alpha \pm \beta$. Hence, if $\lambda > 0$, $\lambda \neq \lambda_k$, we have

$$\begin{aligned} \mu_1(\lambda) &= \alpha - \beta = -\sqrt{\lambda}\tan\frac{\sqrt{\lambda}L}{2}, \\ \mu_2(\lambda) &= \alpha + \beta = \sqrt{\lambda}\cot\frac{\sqrt{\lambda}L}{2}. \end{aligned} \tag{3.5}$$

If $\lambda < 0$ we get by a similar calculation (or using (3.5) with complex arguments)

$$\begin{aligned} \mu_1(\lambda) &= \alpha - \beta = \sqrt{-\lambda}\tanh\frac{\sqrt{-\lambda}L}{2}, \\ \mu_2(\lambda) &= \alpha + \beta = \sqrt{-\lambda}\coth\frac{\sqrt{-\lambda}L}{2}. \end{aligned} \tag{3.6}$$

Note that $\mu_1(\lambda)$ and $\mu_2(\lambda)$ are differentiable at $\lambda = 0$. Figure 3.1 shows μ_1 and μ_2 as functions of λ. We can now summarise the properties of e^{-D_λ}.

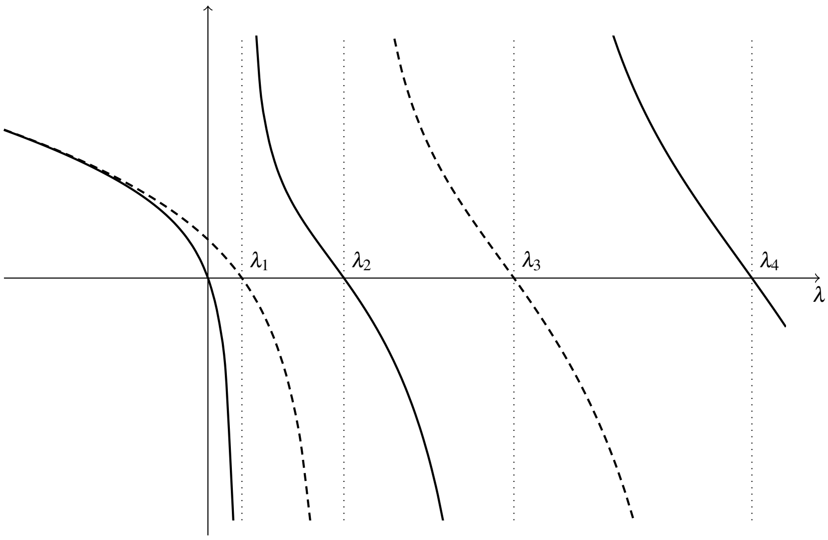

Fig. 3.1 Plot of $\mu_1(\lambda)$ (solid) and $\mu_2(\lambda)$ (dashed).

Theorem 3.1 *Let D_λ be the Dirichlet-to-Neumann operator as defined above. Suppose that $\lambda_k = (k\pi/L)^2$, $k \in \mathbb{N}$, are the eigenvalues of $-u'' = \lambda u$ with Dirichlet boundary conditions. Then the following assertions are true.*

(i) *e^{-tD_λ} is positive if and only if $\lambda < \lambda_1$ or $\lambda_{2k} < \lambda < \lambda_{2k+1}$. Positivity and non-positivity change at every eigenvalue λ_k, $k \in \mathbb{N}$;*

(ii) *We have*

$$\lim_{\lambda\to\lambda_k+} e^{-tD_\lambda} = \frac{1}{2}\begin{bmatrix} 1 & (-1)^k \\ (-1)^k & 1 \end{bmatrix}$$

for all $t > 0$. That limit is a projection, and in particular a degenerate semigroup.

(iii) *$\lim_{\lambda\to\lambda_k-} \|e^{-tD_\lambda}\| = \infty$ for all $k \in \mathbb{N}$.*

Proof (i) From (3.4) the semigroup e^{-tD_λ} is positive if and only if $\sinh\beta t > 0$ for all $t > 0$. By (3.2) this is the case for $\lambda \in (\lambda_{2k}, \lambda_{2k+1})$, $k \in \mathbb{N}$. If $\lambda < 0$ then $\beta(\lambda) > 0$ always by (3.3), so $\sinh\beta t > 0$ for all $t > 0$.

(ii) The entries of the matrix (3.4) are

$$\begin{aligned} e^{-\alpha t}\cosh\beta t &= \frac{1}{2}\big(e^{-t(\alpha-\beta)} + e^{-t(\alpha+\beta)}\big) = \frac{1}{2}\big(e^{-t\mu_1(\lambda)} + e^{-t\mu_2(\lambda)t}\big) \\ e^{-\alpha t}\sinh\beta t &= \frac{1}{2}\big(e^{-t(\alpha-\beta)} - e^{-t(\alpha+\beta)}\big) = \frac{1}{2}\big(e^{-t\mu_1(\lambda)} - e^{-t\mu_2(\lambda)}\big) \end{aligned} \tag{3.7}$$

If k is even, then $\mu_1(\lambda) \to 0$ as $\lambda \to \lambda_k$ and $\mu_2(\lambda) \to \infty$ as $\lambda \to \lambda_k+$. If k is odd, then $\mu_1(\lambda) \to \infty$ as $\lambda \to \lambda_k+$ and $\mu_2(\lambda) \to 0$ as $\lambda \to \lambda_k$. Hence,

$$\lim_{\lambda\to\lambda_k+} e^{-\alpha(\lambda)t}\cosh\beta(\lambda)t = \frac{1}{2} \quad\text{and}\quad \lim_{\lambda\to\lambda_k+} e^{-\alpha(\lambda)t}\sinh\beta(\lambda)t = \frac{(-1)^k}{2},$$

proving (ii).

(iii) If k is even, $\mu_2(\lambda) \to -\infty$ as $\lambda \to \lambda_k-$. If k is odd, then $\mu_1(\lambda) \to -\infty$ as $\lambda \to \lambda_k-$. Hence, at least one term in (3.7) tends to infinity. □

Note that the limiting behaviour of $\mu_1(\lambda)$ and $\mu_2(\lambda)$ observed in Figure 3.1 is valid in general as shown in [5].

4 Non-positivity on the disc

In this section we discuss the Dirichlet-to-Neumann operator on the unit disc in $\mathbb{R}^2$, and in particular we prove Theorem 1.1. We first solve

$$\begin{aligned} \Delta u + \lambda u &= 0 \quad \text{in } B, \\ u &= \varphi \quad \text{on } \partial B, \end{aligned} \tag{4.1}$$

where $B := B(0,1)$ is the unit ball in $\mathbb{R}^2$ and $\lambda > 0$. Given the Fourier series expansion of φ we want to express $D_\lambda\varphi$ and the semigroup $e^{-tD_\lambda}\varphi$ in terms of that Fourier series. By separating variables it turns out that $J_k(\sqrt{\lambda})e^{ik\theta}$, $k \in \mathbb{Z}$, are solutions of $\Delta u + \lambda u = 0$, where J_k are the Bessel functions of the first kind. Note that $J_{-k} =$

$(-1)^k J_k$, so $J_{|k|}(\sqrt{\lambda})e^{\pm ik\theta}$ are solutions for all $k \in \mathbb{N}$; see [8, page 304]. If $J_k(\sqrt{\lambda}) \neq 0$, then

$$u_k(r,\theta) = \frac{J_{|k|}(\sqrt{\lambda}r)}{J_{|k|}(\sqrt{\lambda})}e^{ik\theta} \tag{4.2}$$

is the solution of (4.1) with boundary values $e^{ik\theta}$ on ∂B. The formula is also valid for $\lambda < 0$, where $\sqrt{\lambda} \in \mathbb{C}$. Hence we can apply the Dirichlet-to-Neumann operator

$$D_\lambda e^{ik\theta} = \frac{\partial u_k}{\partial \nu} = \frac{e^{ik\theta}}{J_{|k|}(\sqrt{\lambda})}\frac{d}{dr}J_{|k|}(\sqrt{\lambda}r)\Big|_{r=1} = \frac{\sqrt{\lambda}J'_{|k|}(\sqrt{\lambda})}{J_{|k|}(\sqrt{\lambda})}e^{ik\theta}.$$

To simplify notation we set

$$d_k(\lambda) := \frac{\sqrt{\lambda}J'_{|k|}(\sqrt{\lambda})}{J_{|k|}(\sqrt{\lambda})}, \tag{4.3}$$

so that

$$D_\lambda e^{ik\theta} = d_k(\lambda)e^{ik\theta}. \tag{4.4}$$

We prove properties of $d_k(\lambda)$ as a function of λ and k in Lemma 4.2 below.

If $J_k(\sqrt{\lambda}) = 0$, then $J_k(\lambda)e^{ik\theta}$ is zero on ∂D and therefore λ is an eigenvalue of the Dirichlet Laplacian on B with eigenfunctions $J_k(\sqrt{\lambda}r)e^{\pm ik\theta}$, $k \in \mathbb{N}$. Alternatively we can write them as real valued functions, namely

$$J_k(\sqrt{\lambda}r)(a\cos k\theta + b\sin k\theta),$$

where $k \in \mathbb{N}$ and $a, b \in \mathbb{R}$. If $j_{k,m}$, $m \in \mathbb{N}$ are the positive zeros of J_k, then it turns out that

$$\sigma(-\Delta) = \{j_{k,m}^2 \neq 0 \colon k \in \mathbb{N}, m \in \mathbb{N}\}$$

is the Dirichlet spectrum of $-\Delta$ on the unit disc; see [8, pages 304/305]. The first few eigenvalues can be seen on the graph of the Bessel functions in Figure 4.1. Table 4.1 lists their approximate values and multiplicities. Bourget's hypothesis (1866) states that J_k and J_n have no common zeros (other than zero) if $n \neq k$, a conjecture proved by means of a number theoretic result due to CF Siegel [19]; see [14, page 198] or [22, page 484]. Hence all Dirichlet eigenvalues on the disc are either simple or of multiplicity two. We strictly order the distinct eigenvalues $\lambda_1 < \lambda_2 < \lambda_3 \dots$. The first few can be identified from Figure 4.1. They are listed in Table 4.1 together with their multiplicity; see [1, page 409].

Remark 4.1 Note that (4.4) means that $d_k(\lambda)$ is an eigenvalue of D_λ with eigenfunctions $e^{\pm ik\theta}$. By construction, u_k is an eigenfunction of the Laplacian with Robin boundary conditions. More precisely,

$$\begin{aligned} -\Delta u_k &= \lambda u_k && \text{in } B, \\ \frac{\partial u_k}{\partial \nu} - d_k(\lambda)u_k &= 0 && \text{on } \partial B. \end{aligned} \tag{4.5}$$

Such a relationship between the eigenvalues of D_λ and the Laplacian with Robin boundary conditions always holds; see [5]. Another interpretation is that $d_k(\lambda)$ are

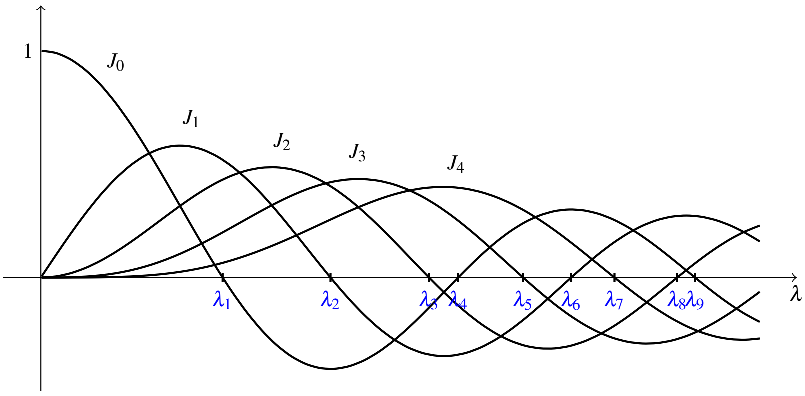

Fig. 4.1 Graphs of the first five Bessel functions $J_0,\dots,J_4$.

Eigenvalue	Approx. Value	Zero of Bessel function	Multiplicity
λ_1	2.4048^2	J_0	1
λ_2	3.8317^2	J_1	2
λ_3	5.1356^2	J_2	2
λ_4	5.5201^2	J_0	1
λ_5	6.3802^2	J_3	2
λ_6	7.0156^2	J_1	2
λ_7	7.5883^2	J_4	2
λ_8	8.4172^2	J_2	2
λ_9	8.6537^2	J_0	1

Table 4.1 First few Dirichlet eigenvalues of $-\Delta$ on B.

the eigenvalues of the Steklov problem for $(\Delta+\lambda I)$-harmonic functions which can be used to describe the trace space of $H^1(\Omega)$; see [6,7]. A graph of these eigenvalues as a function of λ is given in Figure 4.2.

Next we prove some properties of $d_n(\lambda)$.

Lemma 4.2 *The function $d_k(\lambda)$ has the following properties.*

(i) *For all $n\in\mathbb{N}$ and $\lambda\in\mathbb{R}$, $\lambda\neq\lambda_k$, we have*

$$d_n(\lambda)=n-\frac{\sqrt{\lambda}J_{n+1}(\sqrt{\lambda})}{J_n(\sqrt{\lambda})}=n-\sum_{k=1}^{\infty}\frac{2\lambda}{j_{n,k}^2-\lambda}, \tag{4.6}$$

where $j_{n,k}$ is the k-th positive zero of J_n.

(ii) *The function $\lambda\mapsto d_n(\lambda)$ is strictly decreasing between its singularities. Moreover, if $j_{n,k}>0$ is a zero of J_n, then*

$$\lim_{\sqrt{\lambda}\to j_{n,k}^-} d_n(\lambda)=-\infty \qquad \textit{and} \qquad \lim_{\sqrt{\lambda}\to j_{n,k}^+} d_n(\lambda)=\infty.$$

(iii) *For every $L>0$*

$$\lim_{n\to\infty}\big(d_n(\lambda)-n\big)=0 \tag{4.7}$$

uniformly with respect to $\lambda\in(0,L]$.

(iv) *For every* $L>0$ *there exists* $n_0\geq 1$ *such that*

$$n-1<d_n(\lambda)<n$$

for all $n\geq n_0$ *and all* $\lambda\in(0,L]$.

(v) *Let* $\lambda>0$ *and set* $m:=\max\{n\in\mathbb{N}\colon j_{n,1}^2\leq\lambda\}$. *Then* $m\leq\sqrt{\lambda}-j_{0,1}$ *and the map* $n\to d_n(\lambda)$ *is concave for* $n\geq m$.

Proof The first equality in (4.6) is easily obtained from the standard recursion relations between the Bessel functions and their derivatives; see [22, page 45]. The second equality in (4.6) follows from the fact that

$$\frac{sJ_{n+1}(s)}{J_n(s)}=-s\sum_{k=1}^{\infty}\Bigl(\frac{1}{s-j_{n,k}}+\frac{1}{s-j_{n,k}}\Bigr)=\sum_{k=1}^{\infty}\frac{2s^2}{j_{n,k}^2-s^2};$$

see [22, page 498]. Now (ii) follows from (4.6) since for every $k\in\mathbb{N}$ the function

$$\lambda\mapsto\frac{2\lambda}{j_{n,k}^2-\lambda}\tag{4.8}$$

is increasing in λ with asymptote $\lambda=j_{n,k}^2$. To prove (iii) it is sufficient to show that for every $L>0$ the series

$$\sum_{k=1}^{\infty}\frac{2\lambda}{j_{n,k}^2-\lambda}\tag{4.9}$$

converges to zero uniformly with respect to $\lambda\in(0,L]$ as $n\to\infty$. First note that

$$j_{n,k}>j_{0,k}+n\geq j_{0,1}+n\tag{4.10}$$

for all $n\geq 0$ and all $k\geq 1$; see [17]. Now fix $n_0>\sqrt{L}-j_{0,1}$. Since (4.8) is increasing in λ and decreasing in $j_{n,k}$ we conclude from (4.10) that

$$0<\frac{2\lambda}{j_{n,k}^2-\lambda}\leq\frac{2L}{j_{n_0,k}^2-L}$$

for all $n\geq n_0$, $k\in\mathbb{N}$ and $\lambda\in(0,L]$. By the Weierstrass M-test (4.9) converges uniformly with respect to $n\geq n_0$ and $\lambda\in(0,L]$. By (4.10)

$$\frac{2L}{j_{n,k}^2-L}\to 0$$

as $n\to\infty$ uniformly with respect to $k\in\mathbb{N}$. By the uniform convergence of (4.9) it follows that

$$\lim_{n\to\infty}\sum_{k=1}^{\infty}\frac{2\lambda}{j_{n,k}^2-\lambda}=0$$

uniformly with respect to $\lambda\in(0,L]$. Now (iv) immediately follows from (iii). To prove (v) note that $j_{n,k}^2$ is increasing and concave as a function of $n\in\mathbb{N}$ for all $k\geq 1$; see [9]. Thus, (4.9) is the sum of convex functions of $n\geq m$ and therefore convex. Hence (4.6) implies that $d_n(\lambda)$ is a concave function of $n\geq m$. By (4.10) we have $m\leq\sqrt{\lambda}-j_{0,1}$, proving (v). □

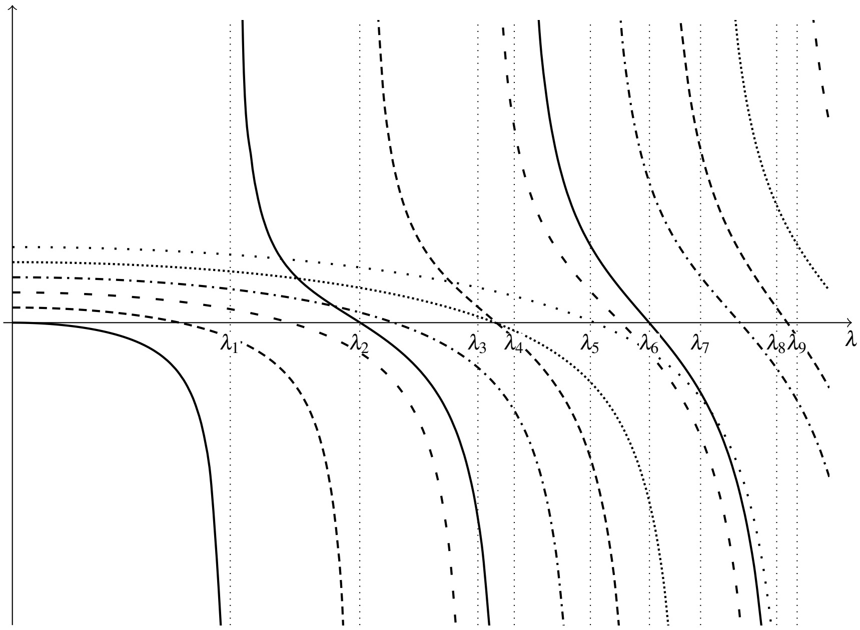

Fig. 4.2 The eigenvalues $d_0(\lambda)$ (solid) and $d_1(\lambda)$ to $d_5(\lambda)$ of D_λ.

We first look at representations of D_λ, e^{-tD_λ} and $a_\lambda(\cdot,\cdot)$ in terms of the Fourier series of the boundary function.

Proposition 4.3 *Suppose that φ and ψ are given by the Fourier series*

$$\varphi = \sum_{k=-\infty}^{\infty} a_k e^{ik\theta} \qquad \textit{and} \qquad \psi = \sum_{k=-\infty}^{\infty} b_k e^{ik\theta}.$$

(i) *The sesquilinear form $a_\lambda \colon H^{1/2}(\partial B) \times H^{1/2}(\partial B) \to \mathbb{C}$ is bounded. If $\varphi, \psi \in H^{1/2}(\partial B)$, then*

$$a_\lambda(\varphi,\psi) = 2\pi \sum_{k=-\infty}^{\infty} a_k \overline{b_k} d_k(\lambda). \tag{4.11}$$

(ii) *If $\varphi \in H^{1/2}(\partial B)$, then*

$$D_\lambda \varphi = \sum_{k=-\infty}^{\infty} a_k d_k(\lambda) e^{ik\theta} \in H^{-1/2}(\partial B). \tag{4.12}$$

Moreover $\sigma(D_\lambda) = \{d_k(\lambda) \colon k \in \mathbb{N}\}$.

(iii) *If $\varphi \in L^2(\partial B)$, then*

$$e^{-tD_\lambda}\varphi = \sum_{k=-\infty}^{\infty} a_k e^{-td_k(\lambda)} e^{ik\theta}. \tag{4.13}$$

Moreover $\sigma(e^{-tD_\lambda}) = \{e^{-td_k(\lambda)} \colon k \in \mathbb{N}\}$, and $e^{\pm ik\theta}$ are the corresponding eigenfunctions.

Proof (i) As $e^{ik\theta}$, $k \in \mathbb{Z}$, is an orthogonal system on ∂B it is easily seen that (4.2) forms an orthogonal system in $H^1(B)$. Using the definition (2.1) of the form $a_\lambda(\cdot,\cdot)$ we get

$$a_\lambda(e^{ik\theta}, e^{i\ell\theta}) = 0$$

if $k \neq \ell$. Since all functions and the domain involved are smooth we can compute $a_\lambda(e^{ik\theta}, e^{ik\theta})$ by means of (2.2). Hence

$$a_\lambda(e^{ik\theta}, e^{ik\theta}) = \int_{\partial B} e^{-ik\theta} D_\lambda e^{ik\theta}\, d\sigma = d_k(\lambda) \int_0^{2\pi} e^{-ik\theta} e^{ik\theta}\, d\theta = 2\pi d_k(\lambda)$$

for all $k \in \mathbb{Z}$. By assumption $\varphi \in H^{1/2}(\partial B)$ which means that

$$\sum_{k=-\infty}^{\infty} |k||a_k|^2 \le \|\varphi\|^2_{H^{1/2}} < \infty,$$

and similarly for the Fourier coefficients b_k of ψ. Let

$$\varphi_n := \sum_{k=-n}^{n} a_k e^{ik\theta} \qquad \text{and} \qquad \psi_n := \sum_{k=-n}^{n} b_k e^{ik\theta}.$$

By Lemma 4.2 we have $d_k(\lambda) \sim k$, so there exists a constant $C > 0$ such that

$$\begin{aligned} |a_\lambda(\varphi_n, \psi_n)| &= 2\pi \Big| \sum_{k=-n}^{n} a_k \overline{b_k} d_k(\lambda) \Big| \le 2\pi C \sum_{k=-n}^{n} |k||a_k||b_k| \\ &\le 2\pi C \Big(\sum_{k=-\infty}^{\infty} |k||a_k|^2 \Big)^{1/2} \Big(\sum_{k=-\infty}^{\infty} |k||b_k|^2 \Big)^{1/2} \le 2\pi C \|\varphi\|_{H^{1/2}} \|\psi\|_{H^{1/2}} \end{aligned}$$

for all $n \in \mathbb{N}$. Hence $a_\lambda : H^{1/2}(\partial B) \times H^{1/2}(\partial B) \to \mathbb{C}$ is a bounded sesquilinear form. In particular, (4.11) follows.

(ii) By (i) $\varphi \mapsto a_\lambda(\varphi, \psi)$ is a bounded linear functional on $H^{1/2}(\partial B)$. Hence

$$D_\lambda \varphi_n = \sum_{k=-n}^{n} a_k d_k(\lambda) e^{ik\theta} \to \sum_{k=-\infty}^{\infty} a_k d_k(\lambda) e^{ik\theta}$$

in $H^{-1/2}(\partial B)$ as $n \to \infty$.

(iii) To get the Fourier series of the semigroup note that the solution of

$$\dot{v} + D_\lambda v = 0, \qquad v(0) = e^{ik\theta}$$

is given by $v(t) = e^{-tD_\lambda} e^{ik\theta} = e^{-td_k(\lambda)} e^{ik\theta}$. As $d_k(\lambda) \sim k$ the terms in the Fourier series (4.13) decay exponentially fast for every $t > 0$, and so the series in (4.13) represents e^{-tD_λ}. □

Remark 4.4 The most obvious way to investigate the positivity of e^{-tD_λ} is to use the fact that the spectral radius of a positive operator has a positive eigenfunction. By the spectral mapping theorem (see [3, Section A-III.6]) that eigenfunction is also the eigenfunction to the minimal eigenvalue of D_λ. The minimal eigenvalue of D_λ is given by

$$d_{k_0}(\lambda) := \min\{d_k(\lambda) \colon k \geq 0\}.$$

That minimum exists since $d_k(\lambda) \to \infty$ as shown in Lemma 4.2. The corresponding eigenspace is spanned by $e^{\pm ik_0\theta}$, or alternatively by $\cos k_0\theta$ and $\sin k_0\theta$. Unless $k_0 = 0$ none of the eigenfunctions is positive. Hence a necessary condition for e^{-tD_λ} to be positive for some or all $t > 0$ is that $d_0(\lambda) \leq d_k(\lambda)$ for all $k \in \mathbb{N}$. We can see in Figure 4.2 how the order of $d_n(\lambda)$ is jumbled by the singularities of $d_k(\lambda)$ at the Dirichlet eigenvalues of $-\Delta$, and that most of the time the eigenvalue $d_0(\lambda)$ having positive eigenfunction is not the smallest one.

Remark 4.5 From Remark 4.4 it is clear that e^{-tD_λ} is not positive if $\lambda > 0$ and $d_0(\lambda) \geq 0$ since by Proposition 2.1 there is always a negative eigenvalue. As $J_0' = -J_1$, the semigroup is therefore not positive between any zero of $J_0(\sqrt{\lambda})$ and the next zero of $J_1(\sqrt{\lambda})$. Note that these zeros are Dirichlet eigenvalues of $-\Delta$. In particular, the semigroup is not positive for $\lambda \in (\lambda_1, \lambda_2)$ and $\lambda \in (\lambda_4, \lambda_6)$; see Figure 4.2. This criterion is far from optimal. It is clear from Figure 4.2 that the semigroup is not positive for $\lambda \in (\lambda_1, \lambda_3)$ and $\lambda \in (\lambda_4, \lambda_8)$.

We note that by (4.13) we have

$$(e^{-tD_\lambda}\varphi)(\theta) = (G_{\lambda,t} * \varphi)(\theta) := \int_{-\pi}^{\pi} G_{\lambda,t}(\theta - s)\varphi(s)\,ds, \tag{4.14}$$

where

$$G_{\lambda,t}(\theta) := \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} e^{-td_k(\lambda)} e^{ik\theta} \tag{4.15}$$

for all $t > 0$. The function $G_{\lambda,t}$ is the "heat kernel" of the analytic semigroup $e^{-tD_k(\lambda)}$ on $L^2(\partial B)$. To prove the positivity of e^{-tD_λ} it is therefore sufficient to show that $G_{\lambda,t} \geq 0$ for all $t > 0$. There is a necessary and sufficient condition for a Fourier series to represent a positive function, namely Herglotz's theorem asserting that the Fourier coefficients of a non-negative function form a positive definite sequence; see [15, Section 7.6]. It is rather difficult to check that a given sequence is positive definite, so we use a representation of $G_{\lambda,t}$ by means of the Fejér kernels

$$K_n(\theta) := \frac{1}{2\pi} \sum_{k=-n}^{n} \Big(1 - \frac{|k|}{n+1}\Big) e^{ik\theta} = \frac{1}{2\pi(n+1)} \left(\frac{\sin\frac{n+1}{2}\theta}{\sin\frac{\theta}{2}}\right)^2 \tag{4.16}$$

which are positive; see [15, page 12].

Proposition 4.6 *For all $t > 0$ and $\lambda > 0$, $\lambda \neq \lambda_k$, we have*

$$G_{\lambda,t}(\theta) = \sum_{n=1}^{\infty} n b_n(\lambda,t) K_{n-1}(\theta), \tag{4.17}$$

where

$$b_n(\lambda,t) := e^{-td_{n+1}(\lambda)} + e^{-td_{n-1}(\lambda)} - 2e^{-td_n(\lambda)}. \tag{4.18}$$

Moreover, $b_n(\lambda,t) \geq 0$ for all $n > \sqrt{\lambda}$ and all $t > 0$.

Proof By Lemma 4.2 $d_n(\lambda) \to \infty$ as $n \to \infty$, so $e^{-td_k(\lambda)} \to 0$ as $k \to \infty$. Moreover, again by Lemma 4.2 there exists $n_0 \geq 0$ such that such that the map $n \mapsto d_n(\lambda)$ is concave for $n \geq n_0$. Hence $k \mapsto e^{-td_n(\lambda)}$ is convex for $n \geq n_0$. Hence the proof of [15, Theorem 4.1] applies and (4.17) follows. □

Using (4.17) we prove that e^{-tD_λ} is eventually positive or even positive in a left neighbourhood of every simple eigenvalue. Let λ_ℓ be a simple eigenvalue of $-\Delta$ on B with Dirichlet boundary conditions. We know that $J_0(\sqrt{\lambda_\ell}) = 0$ and therefore $d_0(\lambda) \to -\infty$ as $\lambda \to \lambda_\ell^-$. Hence there exists $\eta < \lambda_\ell$ and $\mu \in \mathbb{R}$ such that

$$d_0(\lambda) < \mu < \min\{d_k(\lambda) \colon k \in \mathbb{N} \setminus \{0\}\}. \tag{4.19}$$

for all $\lambda \in \Lambda := [\eta, \lambda_\ell)$.

Remark 4.7 Condition (4.19) is satisfied for $\lambda \in (\lambda_3, \lambda_4)$ as numerical calculations or the tabulated values of Bessel functions show; see Figure 4.2 or [1]. One may conjecture that (4.19) is fulfilled for all $\lambda \in (\lambda_{k-1}, \lambda_k)$ if λ_k is a simple eigenvalue. Due to the complicated nature of the curves $d_k(\lambda)$ this is however not clear.

Under condition (4.19) we show that e^{-tD_λ} is positive eventually. By Proposition 4.6 it is sufficient to show that there exists $T \geq 0$ such that the series (4.17) for the kernel $G_{\lambda,t}(\theta)$ is non-negative for all $t > T$ and all $\theta \in [-\pi, \pi]$. According to Proposition 4.6 there exists $m \in \mathbb{N}$ such that $b_n(\lambda,t) \geq 0$ for all $n \geq m$, $\lambda \in \Lambda$ and $t > 0$, where $b_n(\lambda,t)$ is defined by (4.18). Hence to show the positivity of $G_{\lambda,t}(\theta)$ we only need to show that

$$S_m(\lambda,t) := \sum_{n=1}^{m-1} n b_n(\lambda,t) K_{n-1} \geq 0 \tag{4.20}$$

for all $t > T$ and all $\lambda \in \Lambda$. As $0 \leq 2\pi K_{n-1} \leq n$ it is sufficient to show that

$$e^{-td_0(\lambda)} - e^{-td_2(\lambda)} - 2e^{-td_1(\lambda)} - \sum_{n=2}^{m-1} n^2 |b_n(\lambda,t)| \geq 0.$$

for all $t > T$. Equivalently we want

$$1 - e^{-t(d_2(\lambda)-d_0(\lambda))} - 2e^{-t(d_1(\lambda)-d_0(\lambda))} - \sum_{n=2}^{m-1} n^2 |b_n(\lambda,t)| e^{td_0(\lambda)} \geq 0. \tag{4.21}$$

Due to (4.19) there exists $T > 0$ such that (4.21) is valid for all $t \geq T$ and all $\lambda \in \Lambda$. Note that T can be chosen so that strict inequality holds in (4.21), so $G_{\lambda,t}(\theta) > 0$ for all $t > T$ and all $\theta \in [0, 2\pi]$. Hence e^{-tD_λ} is eventually positive and irreducible for all $\lambda \in \Lambda$. To prove the positivity of e^{-tD_λ} for all $t > 0$ for λ in some sub-interval of Λ we note that

$$S_m(\lambda,0) = \sum_{n=1}^{m-1} b_n(\lambda,t) K_{n-1} = 0$$

since $b_n(\lambda,0) = 1+1-2 = 0$ for all $n \in \mathbb{N}$. If we show that $S_m(\lambda,t)$ has a positive derivative for all $t \geq 0$, then (4.20) holds for all $t > 0$. We have

$$\frac{d}{dt}S_m(\lambda,t) = -\sum_{n=1}^{m-1} n\big(d_{n-1}e^{-td_{n-1}} + d_{n+1}e^{-td_{n+1}} - 2d_n e^{-td_n}\big)K_{n-1}. \tag{4.22}$$

We know that

$$M := \max_{1\leq n\leq m} \sup_{\lambda\in\Lambda} |d_n(\lambda)| < \infty.$$

Hence, by (4.19) and since $0 \leq 2\pi K_{n-1} \leq n$ the derivative (4.22) is positive if

$$-d_0(\lambda) - 4Me^{-t(\mu - d_0(\lambda))}\sum_{n=1}^{m} n^2 > 0 \tag{4.23}$$

for all $t > 0$. As $\mu - d_0(\lambda) > 0$ (4.23) is valid for all $t \geq 0$ if it valid for $t = 0$. As $d_0(\lambda) \to -\infty$ as $\lambda \to \lambda_\ell^-$ we can choose $\tilde{\eta}$ so that

$$-d_0(\lambda) - 4M\sum_{n=1}^{m} n^2 > 0$$

for all $\lambda \in \tilde{\Lambda} := [\tilde{\eta}, \lambda_\ell)$. Clearly, $G_{\lambda,t}$ is not only positive, but strictly positive under the above conditions, so the semigroup is irreducible as well.

A second way to investigate the positivity or non-positivity of e^{-tD_λ} is the Beurling-Deny criterion. It asserts that e^{-tD_λ} is positive if and only if $a_\lambda(\varphi^+,\varphi^-) \leq 0$ for all $\varphi \in H^{1/2}(\partial B)$. We use the standard orthogonal basis on $L^2(\partial B)$, namely the functions $\varphi_n(\theta) = \sin n\theta$ for $n \geq 1$. Note that the functions $\cos n\theta$ only differ from $\sin n\theta$ by a phase shift, so do not give any new information.

Proposition 4.8 *The Fourier series of the positive and negative parts of $\varphi_n(\theta) = \sin n\theta$ are*

$$\begin{aligned} \varphi_n^+(\theta) &= \frac{1}{\pi} + \frac{1}{2}\sin n\theta - \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{4k^2-1}\cos 2nk\theta, \\ \varphi_n^-(\theta) &= \frac{1}{\pi} - \frac{1}{2}\sin n\theta - \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{4k^2-1}\cos 2nk\theta. \end{aligned} \tag{4.24}$$

Moreover,

$$a_\lambda(\varphi_n^+,\varphi_n^-) = \frac{2}{\pi}d_0(\lambda) - \frac{\pi}{4}d_n(\lambda) + \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{(4k^2-1)^2}d_{2kn}(\lambda). \tag{4.25}$$

Proof We easily obtain the Fourier series of φ_1^+ by computing the integrals

$$\int_0^\pi \sin\theta \sin k\theta \, d\theta \qquad \text{and} \qquad \int_0^\pi \sin\theta\cos k\theta\, d\theta$$

for $k \in \mathbb{N}$. The Fourier series of φ_n^+ is then given by replacing θ by $n\theta$ in the series for φ_1^+. Note also that $\varphi_n^- = \varphi_n^+ - \varphi_n$, so (4.24) follows. Using Theorem 4.3 we get (4.25). □

Note that the series (4.25) becomes very sparse if n is large, and, except possibly the n-th term, only contains non-trivial even terms. If $a_\lambda(\varphi_n^+,\varphi_n^-)\geq 0$, then the semigroup e^{-tD_λ} is not positive, otherwise nothing can be said.

We can see from (4.25) that e^{-tD_λ} is not positive in a right neighbourhood of every eigenvalue. We just note that the term $d_n(\lambda)$ is positive and dominates the series for λ in a right neighbourhood of every eigenvalue associated with a zero of J_n. Going through all $n\geq 1$ we have all Dirichlet eigenvalues. We further note that e^{-tD_λ} is not positive in a left neighbourhood of every eigenvalue of the Dirichlet Laplacian associated with a zero of J_{2k}, $k\geq 1$. We can see the graph of $\lambda\mapsto a_\lambda(\varphi_n^+,\varphi_n^-)$ in Figure 4.3. As seen already in Figure 4.2, we see again that e^{-tD_λ} can only be positive in a left neighbourhood of every eigenvalue associated with a zero of J_0.

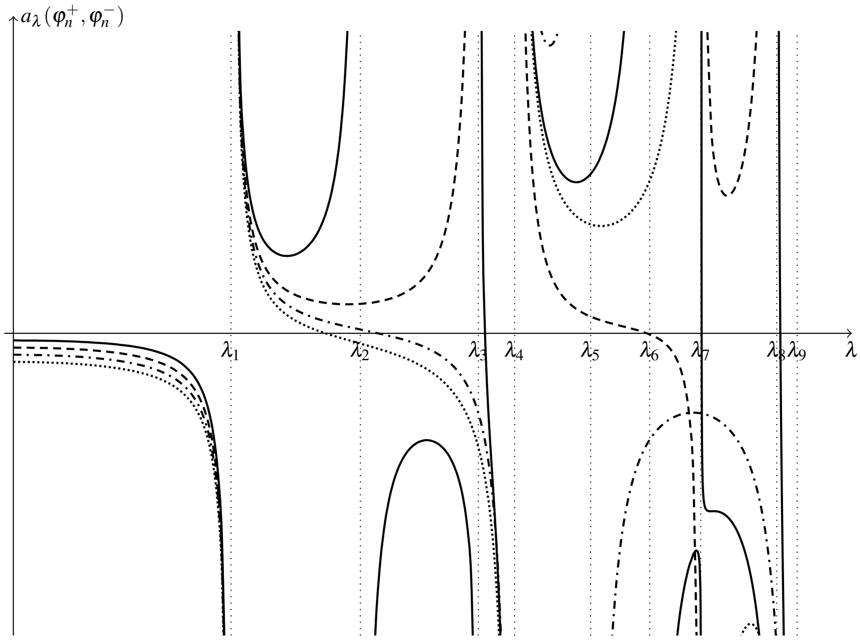

Fig. 4.3 The functions $\lambda\mapsto a_\lambda(\varphi_n^+,\varphi_n^-)$ for $n=1,\dots,4$.

5 Remarks on general domains

In this section we provide reasons for the points made at the end of the introduction. We start by reviewing some features we observed in the example of the ball in Section 4. We first note that the eigenfunctions of D_λ are the traces of the eigenfunctions

u_n of the Steklov (or Robin) eigenvalue problem

$$\begin{aligned} \Delta u_n + \lambda u &= 0 && \text{in } \Omega, \\ \frac{\partial u_n}{\partial \nu} &= d_n(\lambda) u_n && \text{on } \partial\Omega. \end{aligned} \tag{5.1}$$

The corresponding eigenvalues of D_λ were $d_n(\lambda)$. For every n, the function $\lambda \mapsto d_n(\lambda)$ had countably many strictly increasing branches separated by vertical asymptotes at Dirichlet eigenvalues of $-\Delta$ as seen in Figure 4.2. The eigenfunctions of D_λ are the traces $\varphi_n := u_n|_{\partial\Omega}$. In case of the disc these eigenfunctions are independent of λ, but for general domains we do not expect that.

Let Ω be a bounded domain with smooth boundary, and let $0 < \lambda_1 < \lambda_2 < \dots$ be the distinct Dirichlet eigenvalues of $-\Delta$. If $\lambda \neq \lambda_k$ we know that D_λ is a self-adjoint operator on $L^2(\partial\Omega)$ with compact resolvent. Hence there exists an orthonormal basis of eigenfunctions $(\varphi_{\lambda,n})$ of D_λ in $L^2(\partial\Omega)$ corresponding to the eigenvalues $d_n(\lambda)$. From general operator theory $d_n(\lambda) \to \infty$ as $n \to \infty$; see [20, Theorem 5.1]. It is not a priori clear how to order these eigenvalues in a natural way. In case of the disc the order was given by the order of the eigenfunctions $\varphi_{\lambda,n}$ which were independent of λ. The independence of λ is too much to hope for in general, but for each n there should be a continuous family of eigenfunctions $u_{\lambda,n}$ of (5.1) including the Dirichlet eigenfunction where $d_n(\lambda) \to \pm\infty$. The corresponding eigenvalues $d_n(\lambda)$ should form a family of curves similar to a bifurcation diagram with bifurcations from infinity at some of the Dirichlet eigenvalues of $-\Delta$ as shown in Figure 5.1. The corresponding eigenfunctions should form a continuous family across the singularities of $d_n(\lambda)$, such that the nodal line struction on $\partial\Omega$ is preserved. Finally, the totality of curves $d_n(\lambda)$ should be a superposition of many diagrams like the one in Figure 5.1, as can be seen in Figure 3.1 for the interval and in Figure 4.2 for the disc. How exactly they interlace depends on the shape of Ω. The existence of the curves approaching asymptotes at the Dirichlet eigenvalues from the left is proved in [4], but nothing is said about right limits. For the disc there was precisely one branch where the eigen-

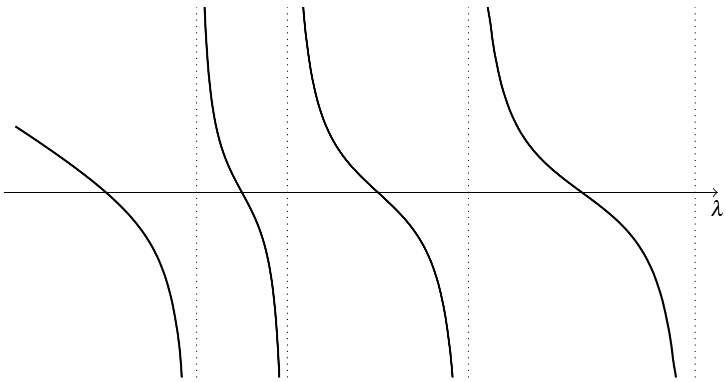

Fig. 5.1 Curve of eigenvalues of D_λ as a function of λ.

function φ_0 did not change sign on $\partial\Omega$. The corresponding eigenfunction of (5.1) has empty or closed nodal lines. We expect a similar phenomenon on general domains, at least in a generic situation.

As done for the disc, we can represent every $\varphi \in H^{1/2}(\partial\Omega)$ in terms of the basis $(\varphi_{\lambda,n})$. Similarly, we can represent the bilinear form $a_\lambda(\cdot\,,\cdot)$ and the semigroup e^{-tD_λ} in terms of that basis. In particular, if

$$\varphi = \sum_{n=0}^{\infty} c_n \varphi_{\lambda,n}, \tag{5.2}$$

then

$$D_\lambda \varphi = \sum_{n=0}^{\infty} c_n d_n(\lambda) \varphi_{\lambda,n} \in H^{-1/2}(\partial\Omega) \qquad \text{if } \varphi \in H^{1/2}(\partial\Omega), \tag{5.3}$$

$$e^{-tD_\lambda} \varphi = \sum_{n=0}^{\infty} e^{-td_n(\lambda)} \varphi_{\lambda,n} \qquad \text{if } \varphi \in L^2(\partial\Omega). \tag{5.4}$$

They correspond to the representations (4.12) and (4.13) in case of the ball. Note that φ is the trace of the solution u of (1.1) given by

$$u = \sum_{n=0}^{\infty} c_n u_{\lambda,n}, \tag{5.5}$$

where $u_{\lambda,n}$ is the solution of (1.1) with boundary values $\varphi = \varphi_{\lambda,n}$. The functions $u_{\lambda,k}$ are eigenfunctions of the Robin problem (1.3) with $\mu = d_n(\lambda)$. Alternatively, (5.5) can be viewed as the Steklov series expansion the $(\Delta + \lambda I)$-harmonic functions with boundary value φ as discussed in [6], where the formula is derived for $\lambda < \lambda_1$, but clearly (5.5) works whenever λ is not a Dirichlet eigenvalue.

For the semigroup e^{-tD_λ} to be positive, the dominating term in (5.4) must be positive. The dominating term is the one corresponding to the minimal eigenvalue

$$d_m(\lambda) := \min\{d_k(\lambda) \colon k \in \mathbb{N}\}$$

of D_λ. Hence, for the semigroup e^{-tD_λ} to be positive we need that $\varphi_{\lambda,m} \geq 0$ on $\partial\Omega$. In fact, it is unlikely that the series (5.4) represents a positive function if $\varphi_{\lambda,m}$ is not strictly positive everywhere. If $\varphi \geq 0$, then $c_m > 0$, again unless there is a positive eigenfunction other than $\varphi_{\lambda,n}$. For the semigroup e^{-tD_λ} to be eventually positive it is sufficient that

$$\limsup_{t\to\infty} \Big(c_m - \sum_{\substack{n=0\\ n\neq m}}^{\infty} |c_n| e^{-t(d_n(\lambda)-d_m(\lambda))} |\varphi_{\lambda,n}| \Big) < 1 \tag{5.6}$$

uniformly with respect to positive initial conditions. This uniformity holds in case of the disc, and may well be true in general. The reason is that the Fourier coefficients c_n need to satisfy certain conditions for φ to be non-negative. We could have taken this approach in the example of the ball. Instead we used a representation involving the Fejér kernels which allowed us to prove that the semigroup is positive for all $t > 0$ for some range of λ using only a finite sum. If (5.6) is true, then the operator e^{-tD_λ} is not only positive, but also irreducible for t large. Hence the spectral radius would be simple, and the only eigenvalue with a positive eigenfunction. In particular, $d_m(\lambda)$ would be a simple eigenvalue of D_λ, and the only one with a positive eigenfunction. There are some conclusions we emphasise again:

1. Simplicity of the eigenvalue λ_k does not imply positivity of the corresponding eigenfunction for λ in any neighbourhood of λ_k, and therefore is not a criterion to expect a positive semigroup near that eigenvalue. In fact, the eigenfunctions to every double eigenvalue on the disc change sign. According to [21] or [13, Examples 6.3 and 6.6] there is an arbitrarily small perturbation of the ball that splits these eigenvalues into simple eigenvalue. The corresponding eigenfunctions still change sign.
2. Positivity in many cases implies irreducibility as explained above, and therefore the simplicity of the minimal eigenvalue. Hence it seems more likely that positivity of the semigroup occurs near simple eigenvalues of the Dirichlet Laplacian. However, there is no proof for that because $d_m(\lambda)$ can be a simple eigenvalue of D_λ without being near a simple eigenvalue of the Dirichlet Laplacian.

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