Introduction to vector bundles

Talk plan

§1. Overview and motivation

§2. First definitions and examples

§3. Sections

§4. Basic constructions

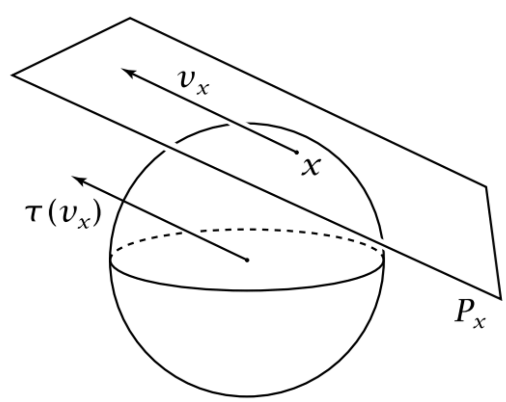
References

§1. Overview and motivation

- X = top. space, "map" = cts function.
- Vector bundles over X arise naturally in many ways.

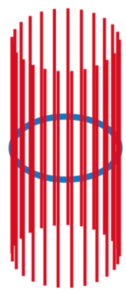
e.g. $X = n$-manifold $\subseteq \mathbb{R}^n$,
$f : X \longrightarrow \mathbb{R}$ a smooth map.

To make sense of orienting X or differentiating f, need TX = tangent bundle $\longrightarrow X$.

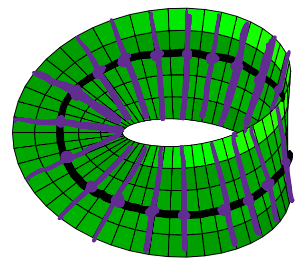


S^2 and its tangent plane at $x \in S^2$ (a translate of the fibre of TS^2 at x).

- <u>Idea</u>: A family of vector spaces (of a fixed dimension) varying continuously and "locally trivially" over the base space X.



Trivial line bundle
$T = S^1 \times \mathbb{R} \longrightarrow S^1$



Möbius bundle
$M \longrightarrow S^1$.

- These are the only line bundles over S^1. Classifying n-dimensional vector bundles over a fixed base X is a hard problem (motivating concepts such as characteristic classes).

- The category of vector bundles on X provides a useful invariant, and gives rise to others,

  e.g. Picard group $(\{\text{line bundles}\}/\cong, \otimes)$

  e.g. the real/complex K-theory of X

better understanding / ways to classify spaces X

tools to prove surprising theorems e.g. classifying f.d. division algebras$/\mathbb{R}$.

- Could say vector bundles$/X$ linearise the structure of X: "linear algebraic topology".

§2. First definitions and examples

- Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.
- Definition: An n-dimensional $\mathbb{F}$-vector bundle is a map

rank

$$p : E \longrightarrow B$$

base space

total space

together with the structure of an $\mathbb{F}$-vector space structure on $p^{-1}(b)$ for all $b \in B$, such that local triviality is satisfied: B is covered by open sets U_i for which there exist homeomorphisms h_i as follows:

local trivialisation

$$\begin{tikzcd}
p^{-1}(U_i) \arrow[r, "\cong", "h_i"'] & U_i \times \mathbb{F}^n \\
p^{-1}(b) \arrow[u, hook] \arrow[r, "\cong", "\text{linear}"'] & \{b\} \times \mathbb{F}^n \arrow[u, hook]
\end{tikzcd}$$

fibre over b

diagram commutes for every $b \in B$.

- <u>Examples</u>: (1) The <u>trivial</u> $\mathbb{F}$–vector bundle of rank n on X is

$$X \times \mathbb{R}^n \xrightarrow{p} X$$

where $p = \pi_1$ is the first projection. We saw a picture for $X = S^1$ above. Local triviality means every vector bundle resembles the trivial bundle over a neighborhood of every point in the base space.

(2) The <u>Möbius</u> bundle is the quotient map

$$\begin{gathered} M = I \times \mathbb{R} / (0,t) \sim (1,-t) \\ \downarrow \\ S^1 = I / 0 \sim 1 \end{gathered}$$

induced by the projection $I \times \mathbb{R} \longrightarrow I$.

(3) The tangent bundle to $S^n \subseteq \mathbb{R}^{n+1}$ may be written as

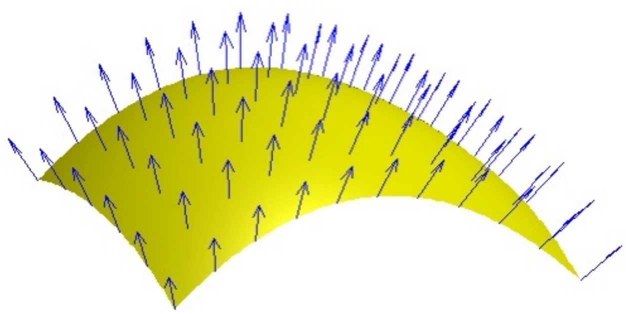
$$\{(x,v)\in S^n\times\mathbb{R}^{n+1} : x\cdot v=0\}$$
$$\downarrow$$
$$S^n.$$

More generally, there is a tangent bundle for every smooth manifold M,

$$TM=\bigsqcup_{x\in M} T_xM \longrightarrow M$$

tangent space at x

(4) In a similar fashion, we can consider the normal bundle $NM\longrightarrow M$.



Note that $NS^n=\{(x,tx)\in S^n\times\mathbb{R}^{n+1} : t\in\mathbb{R}\}$ is globally trivial.

(5) Recall $\mathbb{R}P^n=\{\text{lines through origin in } \mathbb{R}^{n+1}\}$
$$=(\mathbb{R}^{n+1}-\{0\})/\text{scalar multiplication}.$$

Then the canonical or tautological line bundle over $\mathbb{RP}^n$ is

$$\{(l, x) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} : x \in l\},$$

i.e. "the fibre over l is l itself."

Generalisation: Grassmann manifolds [later].

- Definition: Suppose have $\mathbb{F}$-vector bundles

$$E_1 \xrightarrow[p_1]{} X, \qquad E_2 \xrightarrow[p_2]{} X.$$

A vector bundle morphism is a map $h: E_1 \to E_2$ restricting to a linear transformation $p_1^{-1}(x) \to p_2^{-1}(x)$ for all $x \in X$.

- Example: $NS^n \to S^n \times \mathbb{R}$, $(x, tx) \mapsto (x, t)$ is a morphism (in fact, isomorphism).

- Now we have a category $VB(X)$ of vector bundles over X.

§3 Sections

- Definition: A section of a v.b. $p : E \to B$ is a map $s : B \to E$ such that $s(b) \in p^{-1}(b)$ for all $b \in B$, i.e.
$$ps = 1_B.$$

- Every v.b. has a canonical zero section,
$$s_0(b) = 0_b \in p^{-1}(b).$$
This is continuous because we require local trivialisations to be linear in each fibre (and thus to preserve the zero vector).

- Similarly defined functions
$$s_v(b) = v_b \in p^{-1}(b), \quad v \in \mathbb{R}^n$$
need not be cts, and in fact a v.b. may not have everywhere non-zero sections – e.g. $TS^n \to S^n$ for $n \geq 2$ even.

- Exercise: A bundle map
$$\begin{tikzcd} E_1 \arrow[rr, "h"] \arrow[dr, "p_1"'] & & E_2 \arrow[dl, "p_2"] \\ & B & \end{tikzcd}$$
sends

the (image of the) zero section of p_1 to the zero section of p_2.

- In particular, a bundle isomorphism induces homeomorphisms of zero sections — and their complements.
- Example: Trivial $T = S^1 \times \mathbb{R} \longrightarrow S^1$ has disconnected zero section complement; Möbius $M \longrightarrow S^1$ does not. So $M \not\cong T$ as vector bundles.

- Prop: Let $p: E \longrightarrow B$ of rank n. TFAE:
  (1) p is trivial.
  (2) p has n sections $s_1, \dots, s_n$ such that $s_1(b), \dots, s_n(b)$ are linearly indep. in each fibre $p^{-1}(b)$.

Proof: (1) $\Rightarrow$ (2) is clear, just by choosing a basis and taking constant sections.

(2) $\Rightarrow$ (1): Given $s_1, \dots, s_n$, define

$$f: B \times \mathbb{R}^n \longrightarrow E, \quad f(b, t_1, \dots, t_n) = \sum t_i s_i(b).$$

Then $f: \{b\} \times \mathbb{R}^n \longrightarrow p^{-1}(b)$ is a linear iso. in each fibre. Further, $B \times \mathbb{R}^n$ is covered by subsets $U \times \mathbb{R}^n$ such that

$$U \times \mathbb{R}^n \xrightarrow{f} p^{-1}(U) \xrightarrow{\cong} U \times \mathbb{R}^n$$

is the continuous map $(b, t_1, \dots, t_n)$

$\longmapsto (b, \sum t_i s_i(b))$, so f is continuous.

Now f is an iso by the following technical result.

- **Lemma:** A bundle map $h: E_1 \longrightarrow E_2$ is an isomorphism iff a (linear) iso. in each fibre.

Proof: See Lemma 1.1 in Hatcher.

- **Example:** $TS^1 \longrightarrow S^1 = \{(x,y) \in \mathbb{R}^2 : x^2+y^2=1\}$ has the non-zero section

$$(x,y) \longmapsto (-y, x),$$

so $TS^1 \cong$ trivial.

§4 Basic constructions

- <u>Restriction:</u> Given a v.b. $p: E \to B$, and a subspace $A \subseteq B$,
$$p|_A : p^{-1}(A) \longrightarrow A \text{ is a v.b.}$$
- <u>Products:</u> Given v.b. $p_1: E_1 \longrightarrow B_1$, $p_2: E_2 \longrightarrow B_2$, have a v.b. $p_1 \times p_2: E_1 \times E_2 \longrightarrow B_1 \times B_2$.
- <u>Direct sums:</u> Suppose $E_1 \xrightarrow{p_1} B \xleftarrow{p_2} E_2$ v.b. over a common base space.

Then for $\Delta = \{(b,b) \in B \times B : b \in B\}$, have
$$\begin{array}{c}(p_1 \times p_2)|_\Delta : (p_1 \times p_2)^{-1}(\Delta) = \{(e_1,e_2) : p_1(e_1) = p_2(e_2)\} \\ \downarrow \\ B,\end{array}$$
which we denote $E_1 \oplus E_2 = E_1 \times_B E_2$:
$$\begin{tikzcd} E_1 \times_B E_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & E_2 \arrow[d, "p_2"] \\ E_1 \arrow[r, "p_1"] & B \end{tikzcd}$$

• **Examples:** (1) trivial $\oplus$ trivial $\cong$ trivial.

(2) $TS^n \oplus NS^n = \{(x, v, tx)\} \subseteq S^n \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$

$\cong$ trivial $= S^n \times \mathbb{R}^{n+1}$

by $(x, v, tx) \longmapsto (x, v + tx)$.

Thus $TS^n \oplus$ trivial $=$ trivial (TS^n is said to be stably trivial)

• **Gluing:** Let $p : E \longrightarrow B$, with local triv.

$$h_i : p^{-1}(U_i) \xrightarrow{\cong} U_i \times \mathbb{R}^n.$$

Then $$E \cong \left(\bigsqcup_i U_i \times \mathbb{R}^n \right) / \sim$$

where $(x, v) \in U_i \times \mathbb{R}^n \sim h_j h_i^{-1}(x, v)$ for all $x \in U_i \cap U_j$.

Observe: $h_j h_i^{-1}(x, v) = (x, g_{ji}(x)(v))$

some $g_{ji}(x) \in GL(\mathbb{R}^n) = GL_n(\mathbb{R})$.

gluing function $U_i \cap U_j \to GL_n(\mathbb{R})$

Can show g_{ji} satisfy cocycle condition:

$$(*) \quad g_{kj}\, g_{ji} = g_{ki} \quad \text{on } U_i \cap U_j \cap U_k.$$

Conversely, any collection of gluing functions

$$g_{ji} : U_i \cap U_j \longrightarrow GL_n(\mathbb{R})$$

satisfying $(*)$ can be used to construct a v.b. $E \longrightarrow B$.

- Tensor products: Given $p_1 : E_1 \longrightarrow B$,
  $p_2 : E_2 \longrightarrow B$,

  Choose a common trivialisation for p_1, p_2
  $\rightsquigarrow$ gluing functions g^1_{ji}, g^2_{ji}.

  Then have

$$g^1_{ji} \otimes g^2_{ji} : U_i \cap U_j \longrightarrow GL(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) = GL(\mathbb{R}^{n_1 n_2}),$$

  gluing functions for rank $n_1 n_2$ bundle $E_1 \otimes E_2 \longrightarrow B$.

• <u>Definitions:</u> (1) Let $\mathcal{U}$ denote an open cover of a space X. A collection of maps $\varphi_i : X \to [0,1]$ is said to be a <u>partition of unity subordinate to $\mathcal{U}$</u> in case $\forall i \; \exists U_i$ s.t.

$$\operatorname{supp} \varphi_i \subseteq U_i \quad \text{and} \quad \sum_i \varphi_i = 1$$

assumed finite sum at all $x \in X$

(2) A Hausdorff X is said to be <u>paracompact</u> if every $\mathcal{U}$ admits a subordinate partition of unity.

• <u>Example:</u> Every compact Hausdorff space is paracompact (by Urysohn's lemma).

• <u>Inner products:</u> An inner product on real v.b $p : E \to B$ is a map $I : E \oplus E \to \mathbb{R}$ which is such that

$$I : p^{-1}(b) \times p^{-1}(b) \longrightarrow \mathbb{R}$$

is an $\mathbb{R}$-vector space inner product.

- **Theorem:** If B is paracompact, then every real v.b. $p: E \to B$ admits an inner product.

Proof: (1) Doable for trivial bundle

$$E = U \times \mathbb{R}^n \longrightarrow \mathbb{R}^n, \quad U \subseteq X \text{ open:}$$

$$E \oplus E = U \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$$

$$(u, x, y) \longmapsto \langle x, y \rangle \text{ works.}$$

(2) Let $\mathcal{U}$ be an open cover trivialising p. By (1), can get $\langle -, - \rangle_U$ on $p^{-1}(U)$ for all $U \in \mathcal{U}$. Take subordinate partition of unity $\{\varphi_i\}$ and define

$$E \oplus E = \{(v, w) : p(v) = p(w)\} \longrightarrow \mathbb{R},$$

$$\langle v, w \rangle = \sum_i \varphi_i(p(v)) \langle v, w \rangle_{U_i}.$$

References (See website for details.)

- A. Hatcher. Vector Bundles and K-theory, Ch. 1.1.
- O. Randal-Williams. Characteristic classes and K-theory.

Image credits (order of appearance):

(1) A. Hatcher
(2) Wikimedia Commons user: Oleg Alexandrov.
(3) Wikimedia Commons User: Lantonov.
(4) Wikimedia Commons user: Oleg Alexandrov