Talk – Part I

§1 Context and Motivation

- Fix k alg. closed of char. $p > 0$, G connected, simply connected semisimple alg. group $/k$, $(R \subseteq X, R^\vee \subseteq X^\vee)$ the associated root system.

- Recall: After choosing $R^+ \subseteq R$, the dominant weights $\Lambda = X^+$ index the iso-classes of simples in $\operatorname{Rep} G$, say $L(\lambda)$.

- Consider $X_1 = \{\lambda \in X^+ : 0 \leq \langle \alpha^\vee, \lambda \rangle < p$ for $\alpha \in R^+\}$,

  "dominant weights with ≤ 1 p-adic digit."

- Steinberg's tensor theorem: If $\lambda = \sum_{i=0}^{m} p^i \lambda_i$ with all $\lambda_i \in X_1$, then

$$L(\lambda) \simeq L(\lambda_0) \otimes L(\lambda_1)^{[1]} \otimes \cdots \otimes L(\lambda_m)^{[m]}$$

where $[i]$ denotes the i-th Frobenius twist.

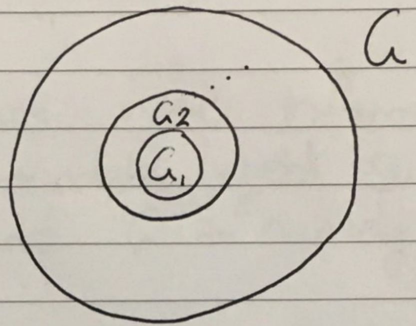
$$\Downarrow$$

- Knowing about $L(\lambda)$ for $\lambda \in X_1$ $\Longleftrightarrow$ knowing about $L(\lambda)$ for all $\lambda \in X^+$.

- <u>Recall also:</u> Iterates of $Fr: G \longrightarrow G^{(1)}$ have kernels

$$G_n = \ker Fr^n \subseteq G,$$

infinitesimal group subschemes of G.



- Let $\mathfrak{g} = \mathrm{Lie}\, G = T_1 G$ = left-invariant derivations $D: k[G] \longrightarrow k[G]$, i.e.

$$\begin{tikzcd} k[G] \arrow[r, "Fr_a"] \arrow[d, "D"'] \arrow[dr, phantom, "\circlearrowleft"] & k[G] \otimes k[G] \arrow[d] \\ k[G] \arrow[r, "\Delta"] & k[G] \otimes k[G] \end{tikzcd}$$

Then bracket on $\mathfrak{g}$ = commutator of derivations, and it is a p-Lie algebra with $X^{[p]} = X^p$. Have

$$u(\mathfrak{g}) = U(\mathfrak{g}) / X^p - X^{[p]}.$$

- In char. 0, have $\mathrm{Rep}(G) \xrightarrow{f.f.} \mathrm{Rep}(\mathfrak{g})$

But in char. p, just have

$$\mathrm{Rep}(G) \hookrightarrow \varinjlim \mathrm{Rep}(G_n)$$

with $\mathrm{Rep}(G_1) \simeq u(\mathfrak{g})\text{-mod}$ (via $k[G_1] \cong u(\mathfrak{g})^*$).

- <u>Moral:</u> Understanding reps of Frobenius kernels aids understanding of $\mathrm{Rep}(G)$.

- Concrete example: Curtis' theorem: For $\lambda \in \mathcal{X}_1$, $L(\lambda)|_{G_1}$ is irreducible, and all irreducible G_1-modules are of this form.

So problem of simple chars for G_1 $\Rightarrow$ answer for simple chars of G.

- Today: A case where information on multiplicities for G_1T-modules yields such information for G_1 and G — namely an "independence of p" result.

§2 Overview of results

- Important feature: $\dim_k u(\mathfrak{g}) < \infty \Rightarrow u(\mathfrak{g})$ is the direct product of indecomposable algebras (blocks).

- Let $\mathcal{B}$ = block of k (trivial module). Now Humphreys (1971) shows: if $p > h$, then the simple modules belonging to (not annihilated by) $\mathcal{B}$ are parametrised by W.

- Here h - Coxeter number of $R = \left| \prod_{\substack{\sigma \in W \\ \text{simple}}} \sigma \right|$.

- Now Cartan matrix $M_{\mathcal{B}} = (m_{ww'})$ has size $|W| \times |W|$, where

$m_{ww'}$ = mult. of simple assoc. to w in a comp. series for PIM assoc. to w'.

- Result: For $p \gg 0$, $M_{\mathcal{B}}$ is independent of p.

- Analogues of Vermas: Introduced by Humphreys to aid in proof of linkage principle:

belonging to B
$$\begin{cases} Z_w = Z_{w\cdot 0} = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} k_{w\cdot 0}, & w \in W. \\ \quad \downarrow ! & \\ L_w \quad \text{simple.} & \end{cases}$$

He shows: If Q_w is the projective cover of L_w, then it has a filt. by $Z_{w'}$ factors and

$$(Q_w : Z_{w'}) = [Z_{w'} : L_w] \quad \begin{pmatrix} \text{BH} \\ \text{reciprocity} \end{pmatrix}$$
$$=: d_k(w', w).$$

- Result 2: For all $w, w' \in W$, there is $d(w', w) \in \mathbb{N}$ such that $d_k(w', w) = d(w', w)$ whenever $p \gg 0$.

- Recall: Weyl module $V(\mu) = H^0(-w_0\mu)^*$ for highest weight $\mu \in X_+$.

$$\rightsquigarrow \quad b_k(\mu, \lambda) = [V(\mu) : L(\lambda)].$$

- Linkage principle + translation principle: $b_k(\mu, \lambda)$ det. by the

dominant

$$b_k(w, w') = b_k(w' \cdot 0, w \cdot 0), \quad w \in W_{aff} = W \ltimes \mathbb{Z}R.$$

- Result 3: For all $w, w' \in W_{aff}$, $w\cdot 0, w' \cdot 0 \in X_+$, there is $b(w, w') \in \mathbb{N}$ such that $b_k(w, w') = b(w, w')$ whenever $p \gg 0$.

- <u>Remark:</u> G_1T-modules $\leftrightarrow$ G_1-modules with a compatible $X(T)$-grading
$\leftrightarrow$ $\mathcal{U}(\mathfrak{g})$-modules with a compatible X-grading.

Compatible means: if $M = \bigoplus_{\nu \in X} M_\nu$, then

$$E_\alpha : M_\nu \longrightarrow M_{\nu+\alpha}, \quad H \curvearrowright M_\nu \text{ by } d\nu(H).$$

- Have analogous Vermas, simples $\hat{Z}(\lambda)$, $\hat{L}(\lambda)$
$\rightsquigarrow$ multiplicities $d_k'(w', w) = d_k'(w'\cdot 0, w\cdot 0)$.

- <u>Result 4:</u> For all $w, w' \in W_{aff}$, there is an integer $d'(w', w) \in \mathbb{N}$ such that $d_k'(w', w) = d'(w, w')$ for $p \gg 0$.

- <u>Remark:</u> (1) All of these results have analogues for characteristic 0 theory, stated in terms of

U_p = quantised enveloping algebra at a p-th root of unity.

(2) Conjectured first by Verma (Conjecture V).

§3 Method Outline

- Take $\hat{Q}(\lambda) \longrightarrow \hat{L}(\lambda)$ a projective cover of G_1T-mods. Again $(\hat{Q}(\lambda) : \hat{Z}(\mu)) = [\hat{Z}(\mu) : \hat{L}(\lambda)]$,

$\rightsquigarrow$ Result 4 can be rephrased in terms of $\mathrm{ch}\,\hat{Q}(w\cdot 0)$

- <u>Know:</u> $\hat{Q}_p = \hat{Q}(w_0 \cdot 0)$ has a filtration by exactly one copy of every $\hat{Z}(w_0 \cdot 0)$.

- Have wall-crossing functors Θ_s on the block of $\mathcal{O}$. Write

$$\hat{Q}_k \xrightarrow[\text{of WC functors}]{\text{apply sequence } I} \hat{Q}_{k,I}.$$

- Fact: There is a finite family $\mathcal{I}$ of these sequences such that knowing the decompositions of all $\hat{Q}_{k,I}$ with $I \in \mathcal{I}$ reveals all multiplicities $d_k'(w,w')$.

- Hence the crux of the matter is Theorem A:
There is a $\mathbb{Z}$-algebra $\mathcal{E}$, finitely generated as $\mathbb{Z}$-module, such that

$$(1) \quad \mathcal{E} \otimes_{\mathbb{Z}} k \cong \mathrm{End}_{(G,T)_k}\Big(\bigoplus_{I \in \mathcal{I}} \hat{Q}_{k,I}\Big)$$

for all k, and

$$(2) \quad \mathcal{E} = \bigoplus_{I,J \in \mathcal{I}} \mathcal{E}_{I,J}, \quad \mathcal{E}_{I,J} \otimes_{\mathbb{Z}} k \cong \mathrm{Hom}(\hat{Q}_{k,I}, \hat{Q}_{k,J}).$$

- $A \Rightarrow R4 \Rightarrow R3, R2$ (how?).
- So need: characteristic-free approach to End algebras.
Solution: generalise from G,T-modules.
- Def: Let $U^0 = S(\mathfrak{h}) \cong \mathcal{U}(\mathfrak{h}) \subseteq \mathcal{U}(\mathfrak{g}) = U$ and suppose we have a Noetherian commutative U^0-algebra

$$\pi : U^0 \longrightarrow A.$$

We define a category $\mathcal{C}_A$:
obj: $U \otimes A$-module M, f.g. over A, with an X-grading $M = \bigoplus M_\nu$ satisfying
(1) The M_ν are A-submodules,
(2) $E_\alpha : M_\nu \to M_{\nu+\alpha}$, F_α ... acts by zero by $\pi(H)$ $\forall H \in \mathfrak{h}$ on M_ν + $\nu(H)$.

- <u>morphisms:</u> graded $\mathcal{U}\otimes A$-module homomorphisms.

Immediately have: kernels (Noetherianity), images, cokernels for homomorphisms.

- <u>Key example:</u> When $A = R$ and

$$\pi : \mathcal{U}^0 \longrightarrow R, \quad \pi(H) = 0 \text{ for all } H \in \mathfrak{h}$$
(augmentation),

then $\mathcal{C}_R = G_1T$-mod.

- <u>Categories $\mathcal{C}_A$ admit:</u> induction, base change, linkage principle, blocks, translation functors, and other homological constructions. We outline part of this story. [Parallel to Jantzen.]

- Induction $\rightsquigarrow$ Verma modules $Z_A(\mu)$, $\mu \in \mathcal{X}$.
  $\rightsquigarrow$ Z-filtrations of objects in $\mathcal{C}_A$.

- Say $\lambda \equiv \mu$ if $\mathrm{Hom}_{\mathcal{C}_A}(Z_A(\lambda), Z_A(\mu)) \neq 0$ or $\mathrm{Ext}^1_{\mathcal{C}_A}(Z_A(\lambda), Z_A(\mu)) \neq 0$.

  $\rightsquigarrow$ partition of $\mathcal{X}$ into blocks, say b.

- <u>Def:</u> $\mathcal{D}_A \subseteq \mathcal{C}_A$ full subcategory admitting a Z-filtration,

  $\mathcal{D}_A(b)$ = full subcat. admitting Z-filtration by factors $Z_A(\mu)$, $\mu \in b$.
  $\cap$
  $\mathcal{C}_A(b)$ = full subcat. which is an image of a (projective) object from $\mathcal{D}_A(b)$.

More generally, Ω = union of blocks $\rightsquigarrow$ $\mathcal{D}_A(\Omega)$, $\mathcal{C}_A(\Omega)$.

- If A is "good", can take $\Omega = W_p$-orbit of $\mathcal{H}$ and get translations $\mathcal{C}_A(\Omega) \xrightarrow[T_\Omega^\Gamma]{} \mathcal{C}_A(\Gamma)$, wall-crossings ...

- One such: $A = S(\mathfrak{h})_{\mathfrak{m}}$, $\mathfrak{m} = (H_\alpha)_{\alpha \text{ simple}}$.
Projectives $\hat{Q}_{k,I}$ lift to $\hat{Q}_{A,I}$, with

$$\mathrm{Hom}_{\mathcal{C}_A}(\hat{Q}_{A,I}, \hat{Q}_{A,J}) \otimes_A k \cong \mathrm{Hom}_{\mathcal{C}_k}(\hat{Q}_{k,I}, \hat{Q}_{k,J}).$$

Now, choose $\mathbb{Z}$-forms $\mathfrak{g}_{\mathbb{Z}}, \mathfrak{h}_{\mathbb{Z}}$ and $S = S(\mathfrak{h}_{\mathbb{Z}})$.
$\Rightarrow S \otimes_{\mathbb{Z}} k = U^\circ$, A = S-algebra.

- Thm B: $\exists$ alg. $\hat{\mathcal{E}}$, f.g. as S-module, such that
(1) $\hat{\mathcal{E}} \otimes_S A \cong \mathrm{End}\left(\bigoplus_{I \in \mathcal{I}} Q_{A,I}\right)$,
(2) $\hat{\mathcal{E}} = \bigoplus_{I,J \in \mathcal{I}} \hat{\mathcal{E}}_{I,J}$ where $\hat{\mathcal{E}}_{I,J} \otimes_S A \cong \mathrm{Hom}_{\mathcal{C}_A}(Q_{A,I}, Q_{A,J})$

$\Rightarrow$ Thm A.

- Why is this easier? $\mathcal{C}_k \rightsquigarrow \mathcal{C}_A$ is a deformation to a "flat family" over $\operatorname{Spec} A$:
(1) $- \otimes_A k : \mathcal{C}_A \to \mathcal{C}_k$ bijection on projective obj$/\cong$
(2) P, Q projective $\Rightarrow \mathrm{Hom}_{\mathcal{C}_A}(P,Q)$, free finite rank$/A$.
(3) $\mathrm{Hom}_{\mathcal{C}_A}(P,Q) \otimes_A k \xrightarrow[\cong]{} \mathrm{Hom}_{\mathcal{C}_k}(P \otimes_A k, Q \otimes_A k)$

- Suffices to describe flat family "up to codim. 1":
if
$$A^\beta = A[H_\alpha^{-1} : \beta \neq \alpha \in R^+] \subseteq A^\emptyset = A[H_\alpha^{-1} : \alpha \in R^+],$$
then $A = \bigcap A^\beta$ and

$$\mathrm{Hom}_{\mathcal{C}_A}(P,Q) = \bigcap_\beta \mathrm{Hom}_{\mathcal{C}_{A^\beta}}(P^\beta, Q^\beta) \subseteq \mathrm{Hom}(P^\emptyset, Q^\emptyset)$$
for $p \gg 0$.

§4 The combinatorial category

• Theorem: Let $\beta \in R^+$, $\lambda \in \mathcal{X}$. Then

$$\mathrm{Ext}^1_{\mathcal{C}_{A^\beta}}(Z^\beta(\lambda), Z^\beta(\beta\uparrow\lambda)) = \begin{cases} 0, & \text{if } \beta\uparrow\lambda = \lambda \\ A^\beta/H_\beta A^\beta, & \text{else,} \end{cases}$$

and the class of $Z^\beta(\beta\uparrow\lambda) \hookrightarrow Q \twoheadrightarrow Z^\beta(\lambda)$ generates iff Q is projective.

• Def: $\Omega = W_p$-orbit in $\mathcal{X}$ $\rightsquigarrow$ combinatorial category $\mathcal{K}(\Omega)$:

objects: $\mathcal{M} = ((M(\lambda))_{\lambda\in\Omega}, (\mathcal{M}(\lambda,\beta))_{\beta\in R^+, \lambda\in\Omega})$

where $\mathcal{M}(\lambda)$ is a f.g. $A^\emptyset$-module and $M(\lambda,\beta) \subseteq \begin{cases} M(\lambda) \oplus M(\beta\uparrow\lambda) \\ M(\lambda) \end{cases}$ is an A^β-submodule.

morphisms: clear.

• Def: Choose for all $\lambda \in \Omega$, $\beta \in R^+$ with $\beta\uparrow\lambda \neq \lambda$ an element $e^\beta(\lambda) \in \mathrm{Ext}^1_{\mathcal{C}_{A^\beta}}(Z^\beta(\lambda), Z^\beta(\beta\uparrow\lambda))$.
Then we have a functor

$$\mathcal{V} = \mathcal{V}_\Omega : \mathcal{C}_A(\Omega) \longrightarrow \mathcal{K}(\Omega),$$

$$\mathcal{V}M(\lambda) = \mathrm{Hom}(Z^\emptyset(\lambda), M^\emptyset)$$

$$\mathcal{V}M(\lambda,\beta) = \mathrm{Hom}(Z^\beta(\lambda), M^\beta) \text{ if } \beta\uparrow\lambda = \lambda,$$

and otherwise if $\beta\uparrow\lambda \neq \lambda$ then write $e^\beta(\lambda)$ as class of $Z^\beta(\beta\uparrow\lambda) \hookrightarrow Q^\beta(\lambda) \twoheadrightarrow Z^\beta(\lambda)$ w/ unique iso. $Q^\beta(\lambda)^\emptyset \cong Z^\emptyset(\beta\uparrow\lambda) \oplus Z^\emptyset(\lambda)$, ~~and~~ take

$$\mathcal{V}M(\lambda,\beta) = \mathrm{im}(\mathrm{Hom}(Q^\beta(\lambda), M^\beta) \hookrightarrow \mathrm{Hom}(Q^\emptyset, M^\emptyset)).$$

- <u>Big theorem:</u> Suppose all $e^{\beta}(x)$ generate over A^{β}. Then $\mathcal{V}$ is fully faithful on projective objects,

$$\mathrm{Hom}_{\mathcal{C}_A(\Omega)}(P, Q) \xrightarrow{\cong} \mathrm{Hom}_{K(\Omega)}(\mathcal{V}P, \mathcal{V}Q).$$

- <u>Punchline:</u> Wanted char-independent description of

$$\mathrm{Hom}(\hat{Q}_{A,I}, \hat{Q}_{A,J}) \cong \mathrm{Hom}(\mathcal{V}\hat{Q}_{A,I}, \mathcal{V}\hat{Q}_{A,J})$$
in $\mathcal{C}_A(W_p \cdot 0)$ in $K(W_p \cdot 0)$.

But $K(W_p \cdot 0)$ admits combinatorial description/form over $\mathbb{Z}$!

- <u>Strategy:</u> Exhibit object/$\mathbb{Z}$ specialising to $\mathcal{V}Q_{A,I}$.

- <u>Def:</u> (1) A <u>Y-category</u> (Y = abelian group) is a category equipped with compatible shift functors

$$M \longmapsto M[\nu], \quad \nu \in Y.$$
Also <u>Y-functor</u>.

(2) If $M \in \mathrm{ob}(\mathcal{C})$, $\mathcal{C}$ an additive Y-category, set

$$\mathrm{End}^{\#}_{\mathcal{C}}(M) = \bigoplus_{\nu \in Y} \mathrm{Hom}_{\mathcal{C}}(M[\nu], M)$$

- If $\mathcal{C}_k(\Omega)$ had a projective generator P, then

$$\mathcal{C}_k(\Omega) \cong (\mathrm{End}\, P)\text{-mod}.$$

But it does have a projective "Y-generator", $P = \hat{Q}_{k,I}$, so $Y = p\mathbb{Z}R$

$$\mathcal{C}_k(\Omega) \cong \mathrm{End}^{\#}_{\mathcal{C}_k(\Omega)}\hat{Q}_{k,I}\text{-mod} = (\mathrm{End}^{\#}_{\mathcal{C}_A(\Omega)}\hat{Q}_{A,I}) \otimes_A k\text{-mod}$$
$$= \mathrm{End}^{\#}_{K(\Omega)}(\mathcal{V}\hat{Q}_{A,I}) \otimes k\text{-mod},$$

and using a $\mathbb{Z}$-form of $K(\Omega)$, can show the latter ring B_k admits a $\mathbb{Z}$-form $B_{\mathbb{Z}}$, $B_k \cong B_{\mathbb{Z}} \otimes_{\mathbb{Z}} k$, depending just on k and I.