The Atiyah–Hirzebruch Spectral Sequence

① Preliminaries and statement

② Construction

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① Preliminaries and statement

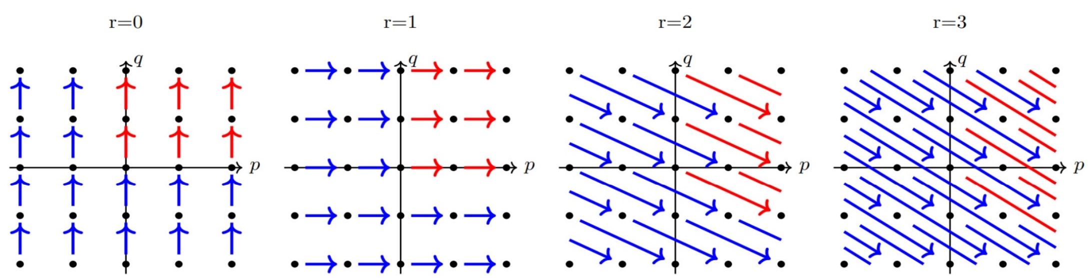
Recall: A cohomology spectral sequence (starting on page $a \in \mathbb{N}$) in an abelian category $\mathcal{A}$ is a family of objects $\{E_r^{pq}\}_{r \geq a}^{p,q \in \mathbb{Z}}$ along with differentials

$$d_r^{pq} : E_r^{pq} \longrightarrow E_r^{p+r,\, q-r+1} \qquad (d_r d_r = 0)$$

and fixed isomorphisms between E_{r+1} and the cohomology of E_r:

$$E_{r+1}^{pq} \cong \frac{\ker d_r^{pq}}{\operatorname{im} d_r^{p-r,\, q+r-1}} \qquad (\text{subquotient of } E_r^{pq})$$

Hence the differentials on page r give complexes lying on a slope of $(1-r)/r$.



In blue, an arbitrary spectral sequence; in red, a first quadrant spectral sequence – $E_r^{pq} \neq 0 \Rightarrow p, q \geq 0$.

Notice: red differentials quickly become zero.
(Need not always happen.)

More generally: E is bounded if there are only finitely many terms $E_a^{pq} \neq 0$ in each total degree $p+q$.

For banded E, fixed p, q, and $r \gg 0$,

$$0 \xrightarrow{d_r^{p-r,\,q+r-1}} E_r^{pq} \xrightarrow{d_r^{pq}} 0 \quad \Rightarrow \quad E_r^{pq} = E_{r+1}^{pq} = \cdots = E_\infty^{pq} \text{ stable value.}$$

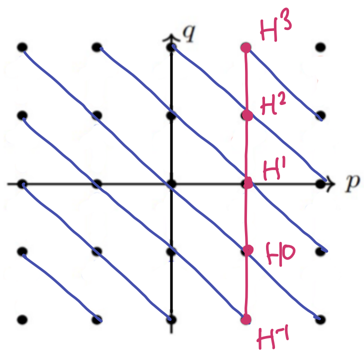
Say E converges to H^*, $E_a^{pq} \Rightarrow H^*$, with $H^n \in \mathcal{A}$, if there is a finite filtration

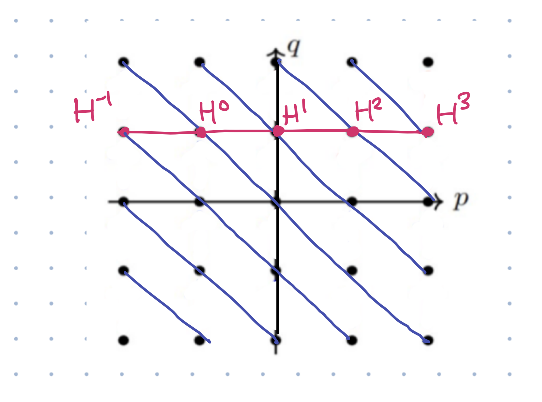
$$0 \subseteq \cdots \subseteq F^{p+1}H^n \subseteq F^pH^n \subseteq \cdots \subseteq H^n$$

such that $F^pH^n / F^{p+1}H^n \cong E_\infty^{pq}$ for $p+q=n$.

Stronger: say E collapses at E_r, $r \geq 2$, if there is exactly one nonzero row or column in E_r^{pq}.

Then $E \Rightarrow H^*$, where $H^n = E_r^{pq}$ for $p+q=n$ and (p,q) on that row or column.





We have similar notions of homology spectral sequences E^r_{pq}.
One advantage commonly enjoyed in cohomology: multiplication.

$$E_r^{pq} \times E_r^{p'q'} \longrightarrow E_r^{p+p',q+q'}$$
$$d_r(xy) = d_r(x)y + (-1)^p x d_r(y). \qquad \text{[Leibnitz]}$$

path connected

Example: Let $f: X \to B$ be a Serre fibration, with fibre F. The (cohomological) Leray–Serre spectral sequence is

$$E_2^{pq} = H^p(B, H^q(F)) \Rightarrow H^{p+q}(X).$$

e.g. $\mathbb{CP}^\infty$

Let us use this to compute $H^*(K(\mathbb{Z},2))$. Consider

$$K(\mathbb{Z},1) \longrightarrow P \longrightarrow K(\mathbb{Z},2) \qquad \text{pathspace fibration}$$

2nd page: $E_2^{pq} = H^p(K(\mathbb{Z},2), H^q(S^1)) = 0$ for $q \neq 0,1$.

E_2

$$\begin{tikzcd}
q & \mathbb{Z} \arrow[drr] & H^1(B) \arrow[drr] & H^2(B) \arrow[drr] & H^3(B) \arrow[drr] & H^4(B) & \\
& \mathbb{Z} & H^1(B) & H^2(B) & H^3(B) & H^4(B) & \cdots \\
& & & & & & p
\end{tikzcd}$$

P contractible $\Rightarrow H^*(P) = \mathbb{Z} \Rightarrow$ only $\mathbb{Z}$ survives on E_∞.
Also: $E_3 = E_\infty$ (differentials too long for $r \geq 3$).

$E_3 = E_\infty$

$$\begin{tikzcd}
q & 0 & 0 & 0 & 0 & 0 & \\
& \mathbb{Z} & 0 & 0 & 0 & 0 & \cdots \\
& & & & & & p
\end{tikzcd}$$

Consequence: displayed maps on E_2 are isomorphisms!
By induction, E_2 simplifies to:

$$
\begin{tikzcd}[row sep=small]
 & & & & & & E_2 \\
q \\
 & \mathbb{Z}a \arrow[drr, "\cong"] & 0 \arrow[drr] & \mathbb{Z}a x_2 \arrow[drr, "\cong"] & 0 \arrow[drr] & \mathbb{Z}a x_4 \arrow[dr] & \\
 & \mathbb{Z} & 0 & \mathbb{Z}x_2 & 0 & \mathbb{Z}x_4 & \cdots \\
 & & & & & & p
\end{tikzcd}
$$

Leibnitz: $d_2(a\overset{x_{2i+2}}{x_{2i}}) = (d_2 a)x_{2i} + a\, d_2(x_{2i}) = \overset{x_2\, x_{2i}}{(d_2 a)x_{2i}}$
thus $H^*(K(\mathbb{Z},2)) = \mathbb{Z}[x_2] \cong \mathbb{Z}[x]$.

Similar computation: Use the Leray–Serre spectral sequence in homology to calculate $H_*(K(\mathbb{Z},2))$.

Atiyah–Hirzebruch (1961): A generalisation of the LSSS!

Let $F \longrightarrow X \xrightarrow{\pi} B$ be a Serre fibration, with B a CW complex, and $h^\bullet$ an additive generalised cohomology theory. Then have

$$E_2^{pq} = H^p(B, h^q(F)) \Rightarrow h^{p+q}(X). \qquad (*)$$

Moreover: the skeletons $B_n \subseteq B \rightsquigarrow$ filtration $X_n \subseteq X$
$\rightsquigarrow$ filtration F of $h^\bullet(X)$,
and convergence in $(*)$ is to $\mathrm{Gr}_F\, h^\bullet(X)$. [Need mild conditions here...]

Note: $h^\bullet$ = singular cohomology recovers LSSS.
The case $F = *$ is what tends to be called the AHSS for $h^\bullet$.
We also have reduced/homology versions of the AHSS.

② Construction

Exact couples: afford a very common way of creating spectral sequences in algebraic topology, due to Massey.

An exact couple in $\mathcal{A}$ is a diagram

$$\begin{tikzcd} D_1 \arrow[rr, "f_1"] & & D_1 \arrow[dl, "g_1"] \\ & E_1 \arrow[ul, "h_1"] & \end{tikzcd}$$

[need not commute!]

[exact at each object]

with $\operatorname{im} f_1 = \ker g_1$, $\operatorname{im} g_1 = \ker h_1$, and $\operatorname{im} h_1 = \ker f_1$

Idea: $(E_1, d_1 = g_1 h_1)$ is "page 1" of a spectral sequence.

How to turn the page?

Let $D_2 = f_1(D_1)$, $E_2 = \ker d_1 / \operatorname{im} d_1$. Then the derived couple is the induced exact couple

restriction

$D_2 = D_1 / \ker f_1$

$$\begin{tikzcd} D_2 \arrow[rr, "f_2"] & & D_2 \arrow[dl, "g_2"] \\ & E_2 \arrow[ul, "h_2"] & \end{tikzcd}$$

in a module category: $g_2(f_1(b)) = \overline{g_1(b)}$

obvious

Then $(E_2, d_2 = g_2 h_2)$ is "page 2", and we obtain further pages by iterated derivation.

These descriptions become literal if $\mathcal{A}$ is a category of $\mathbb{Z}^2$-graded modules over a ring, with

$$\deg(f_1) = (-1, 1), \quad \deg(g_1) = (1, 0), \quad \deg(h_1) = (0, 0).$$

Then $(E_r^{\bullet\bullet}, d_r^{\bullet\bullet})$ is a cohomology spectral sequence.

Convergence Criterion: Suppose that for all $s, t \in \mathbb{Z}$ there exists $c_{st} \in \mathbb{N}$ such that for all $c \geq c_{st}$,

(a) $f_1^{s-c,t+c}$ is zero,

(b) $f_1^{s+c,t-c}$ is an isomorphism.

Consider $L^n = \varprojlim \left(\cdots \to D_1^{p,n-p} \xrightarrow{f_1} D_1^{p-1,n-p+1} \to \cdots \right)$,

Then $E_1^{pq} \Rightarrow L^{p+q}$ ← filtration:

$$F^p L^n = \operatorname{Ker}\left(L^n \longrightarrow D_1^{p-1,n-p+1}\right)$$

Proof: [Kochman, Lemma 2.6.2]

In our case: The exactness axiom for $h^\bullet$ provides a LES

$$\cdots \longrightarrow h^{s+t}(X_s, X_{s-1}) \longrightarrow h^{s+t}(X_s) \longrightarrow h^{s+t}(X_{s-1}) \xrightarrow{\partial} h^{s+t+1}(X_s, X_{s-1}) \longrightarrow \cdots$$

Hence let our exact couple be

$$\begin{tikzcd}
\prod_{s,t} h^{s+t}(X_s) \arrow[rr, "{(-1,1)}"] & & \prod_{s,t} h^{s+t}(X_s) \arrow[dl, "{(1,0)}"] \\
& \prod_{s,t} h^{s+t}(X_s, X_{s-1}) \arrow[ul, "{(0,0)}"] &
\end{tikzcd}$$

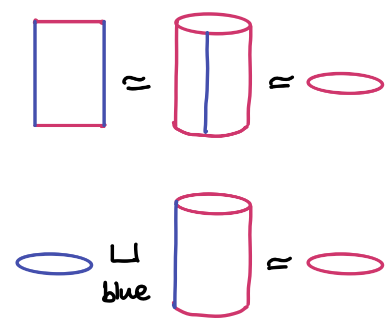
Page 1: For $\sigma \in C(s) = \{s\text{-dimensional cells of } B\}$,

$$\pi^{-1}(\sigma, \partial\sigma) \simeq (D^s, S^{s-1}) \times F$$

← weak homotopy equivalence

$$
\begin{aligned}
\text{Hence } E_1^{st} &= h^{s+t}(X_s, X_{s-1}) \\
&\cong \tilde{h}^{s+t}(X_s/X_{s-1}) \\
&\cong \tilde{h}^{s+t}\Big(\bigvee_{\sigma \in C(s)} S^s \wedge F_+\Big) \\
\text{[wedge axiom]} \quad &\cong \prod_\sigma \tilde{h}^{s+t}(S^s \wedge F_+) \\
\text{[suspension iso]} \quad &\cong \prod_\sigma \tilde{h}^{t}(F_+) \\
&= \prod_\sigma h^{t}(F) \\
\text{[def.]} \quad &= C^s_{\text{cell}}(B, h^t(F)).
\end{aligned}
$$

cellular cochains



E_1

$$
\begin{array}{l}
q \uparrow \\
C^0_{\text{cell}}(B, h^1(F)) \xrightarrow{d_1} C^1_{\text{cell}}(B, h^1(F)) \xrightarrow{d_1} C^2_{\text{cell}}(B, h^1(F)) \\
C^0_{\text{cell}}(B, h^0(F)) \xrightarrow{d_1} C^1_{\text{cell}}(B, h^0(F)) \xrightarrow{d_1} C^2_{\text{cell}}(B, h^0(F)) \\
\hfill \rightarrow p
\end{array}
$$

Page 2: check d_1 = cellular differential (not too hard).
Then cellular cohomology = singular cohomology for B.

The convergence criterion applies if either:
(1) B is finite dimensional, or
(2) $h^\bullet(F)$ is bounded below in degree.

Convergence is then to $L^n = \varprojlim(\cdots \to h^n(X_{p+1}) \to h^n(X_p) \to \cdots)$,
which is $h^n(X)$ in good cases.

These are our "mild conditions".

Technically also needed: F weakly contractible or $\pi_1(B) = 0$.

③ Examples and applications

K-theory: Let $h^\bullet = K^\bullet_{\mathbb{C}}$, $F = *$, X finite dimensional.
It follows from Bott periodicity that

$$K^n = \begin{cases} K^0 & \text{for } n \text{ even,} \\ K^1 & \text{for } n \text{ odd,} \end{cases}$$

i.e. $K^\bullet$ is 2-periodic. In particular,

$$K^n(*) = \begin{cases} \mathbb{Z} & \text{for } n \text{ even,} \\ 0 & \text{for } n \text{ odd.} \end{cases}$$

This allows us to simplify the AHSS. Observe page 2:

$$\begin{array}{c|cccc}
q & & & & \\
 & 0 & 0 & 0 & 0 \\
 & H^0(X,\mathbb{Z}) & H^1(X,\mathbb{Z}) & H^2(X,\mathbb{Z}) & H^3(X,\mathbb{Z}) \\
 & 0 & 0 & 0 & 0 \\
 & H^0(X,\mathbb{Z}) & H^1(X,\mathbb{Z}) & H^2(X,\mathbb{Z}) & H^3(X,\mathbb{Z}) \\
\hline
 & & & & \to p
\end{array}$$

The rows and differentials are periodic

$\Rightarrow$ all content is on row 0 and we can give info more concisely.

AHSS for K-theory: There is a sequence of $\mathbb{Z}$-graded abelian groups E_r^p, $r \geq 1$, and maps $d_r^p : E_r^p \to E_r^{p+r}$, such that:

(1) $E_2^p = H^p(X, \mathbb{Z})$.

(2) $E_{r+1}^p = \dfrac{\ker d_r^p}{\operatorname{im} d_r^{p-r}}$ for all r, p.

(3) $d_r = 0$ and $E_{r+1} = E_r$ for even r.

(4) $E_\infty^p = \dfrac{\ker(K^p(X) \longrightarrow K^p(X_{p-2}))}{\ker(K^p(X) \longrightarrow K^p(X_p))}$

[the groups stabilise for each p, since X is finite dimensional.]

In fact, consider $F^s K^p(X) = \ker(K^p(X) \to K^p(X_s))$.
If $\dim X = n$, then we get

$$K^p(X) = F^{-1}K^p(X) \supseteq F^0 K^p(X) \supseteq \cdots \supseteq F^n K^p(X) = 0 \qquad (F)$$

Bott periodicity $\rightsquigarrow$ restrict attention to $p = 0, 1$.

Lemma: If $s \geq 0$ is even, then

$$F^s K^0(X) \overset{(0)}{=} F^{s+1} K^0(X) \quad \text{and} \quad F^{s-1} K^1(X) \overset{(1)}{=} F^s K^1(X)$$

Proof: Use the K-sequence of (X, X_{s+1}, X_s), as well as

$$K^0(X_{s+1}, X_s) = \tilde{K}^0(\vee S^{s+1}) = \tilde{K}^{-s-1}(\vee S^0) = 0,$$

to see $K^0(X, X_{s+1}) \twoheadrightarrow K^0(X, X_s)$.

This readily gives (0) and (1) is similar.

Hence (F) has the following form (composition factors shown):

$$K^0(X) \overset{E_\infty^0}{\text{———}} F^0K^0(X) \overset{E_\infty^2}{\text{———}} F^2K^0(X) \overset{E_\infty^4}{\text{———}} \cdots \quad (F_0)$$
$$K^1(X) \underset{E_\infty^1}{\text{———}} F^1K^1(X) \underset{E_\infty^3}{\text{———}} F^3K^1(X) \underset{E_\infty^5}{\text{———}} \cdots \quad (F_1)$$

Rational cohomology connection: Recall that $ch_{\mathbb{Q}}$ affords

$$K^0(X)\otimes\mathbb{Q} \cong H^{even}(X), \quad K^1(X)\otimes\mathbb{Q} \cong H^{odd}(X)$$

Hence

$$\operatorname{rank} E_2^* = \operatorname{rank} H^*(X) = \operatorname{rank}(K^0(X)\oplus K^1(X))$$
$$= \operatorname{rank} E_\infty^*.$$

But $\operatorname{rank} E_r^p \geq \operatorname{rank} E_{r+1}^p = \dfrac{\ker d_r^p}{\operatorname{im} d_r^{p-r}}$, with equality

$\Rightarrow \operatorname{rank} E_r^p = \operatorname{rank} \ker d_r^p \Rightarrow \operatorname{im} d_r^p$ torsion.

$\Rightarrow d_r^p \otimes \mathbb{Q} = 0.$

Thus $\underbrace{E_2^* \otimes \mathbb{Q}}_{H^*(X,\mathbb{Q})} \simeq \underbrace{E_\infty^* \otimes \mathbb{Q}}_{(K^0(X)\oplus K^1(X))\otimes\mathbb{Q}}$, agreeing with $ch_{\mathbb{Q}}$.

Complex projective space: Recall $H^*(\mathbb{CP}^n) = \mathbb{Z}[\alpha]/\alpha^{n+1}$,
so that E_2^* looks like deg 2

$$\begin{array}{cccccccl} \mathbb{Z} & 0 & \mathbb{Z} & 0 & \cdots & 0 & \mathbb{Z} & \\ \hline 0 & 1 & 2 & 3 & & 2n-1 & 2n & \to p \end{array}$$

But then odd-degree and even degree differentials are zero!

Hence $E_2^p = E_\infty^p$ and we deduce

$(F_0) \Rightarrow K^0(\mathbb{C}P^n) = \mathbb{Z}^{n+1}$, $(F_1) \Rightarrow K^1(\mathbb{C}P^n) = 0$.

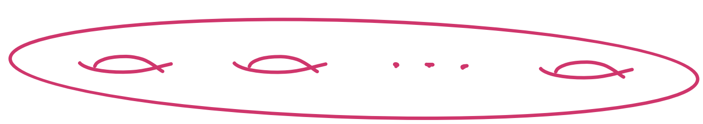
[$\mathbb{Z}$ is projective]

Remark: In fact $K^0(\mathbb{C}P^n) \cong H^*(\mathbb{C}P^n)$ as a ring. Indeed, the AHSS in K-theory is multiplicative, with

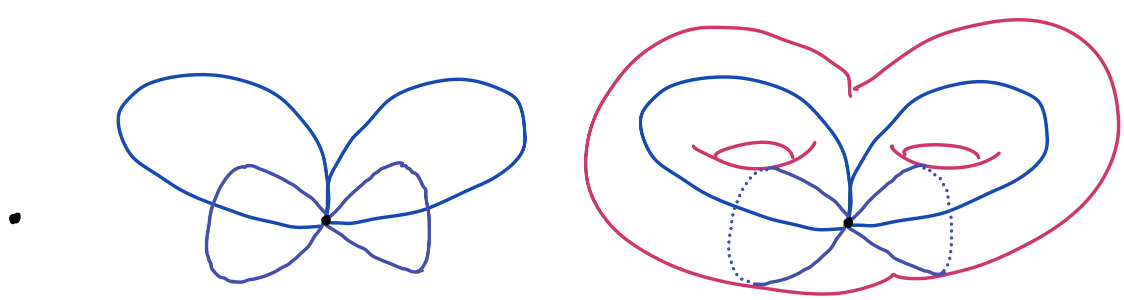
$$\left.\begin{aligned} E_2^* &= H^*(X) \\ E_\infty^* &\cong Gr_F K^*(X) \end{aligned}\right\} \text{ as graded rings.}$$

A generator in degree 2 for $K^0(\mathbb{C}P^n)$ is $\xi - 1$, where ξ is the Hopf (tautological) bundle.

Closed orientable surfaces: Let Σ_g denote a closed orientable surface of genus $g \geq 1$.



This is a CW complex: one 0-cell, $2g$ 1-cells, and one 2-cell.



The cohomology ring $H^*(\Sigma_g)$ has components

$$\begin{array}{cccccc} \mathbb{Z} & \mathbb{Z}^{2g} & \mathbb{Z} & 0 & 0 & \cdots \\ \hline 0 & 1 & 2 & 3 & 4 & \to p \end{array}$$

Now, $d_2 = 0$ and d_3 is already too long to be nonzero, so all differentials are zero!

Thus $K^0(\Sigma_g) = H^{even}(\Sigma_g) = \mathbb{Z}^2$ and
$$K^1(\Sigma_g) = H^{odd}(\Sigma_g) = \mathbb{Z}^{2g}.$$

Finite groups: Let G be a finite group, M a G-module. Recall group cohomology:

$$M^G = \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, M) = H^0(G, M) \rightsquigarrow H^n(G, M) = \mathrm{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z}, M).$$

[G-invariants in M] [Derived functors]

The classifying space BG is a colimit of finite CW complexes X_n and $H^*(BG, \mathbb{Z}) = H^*(G, \mathbb{Z})$.

Also, let $R(G) = [\mathrm{Rep}_{\mathbb{C}}(G)]$ be the representation ring (or character ring) of G over $\mathbb{C}$.

It has an augmentation map $\varepsilon : R(G) \to \mathbb{Z}$ with
$\varepsilon(M) = \dim M$, $M \in \mathrm{Rep}_{\mathbb{C}}(G)$.

Let $I = \ker \varepsilon \trianglelefteq R(G)$ and $\widehat{R(G)} = \varprojlim_m R(G)/I^m$ the I-adic completion.

Let $\xi : EG \longrightarrow BG$ be the universal G-bundle.

$\rightsquigarrow$ principal G-bundles ξ_n on X_n.

Given $\rho : G \longrightarrow GL_n(\mathbb{C})$, have $\rho(\xi_n)$ a principal $GL_n(\mathbb{C})$-bundle, i.e. vector bundle, on X_n.

$\rightsquigarrow$ compatible maps $\alpha_n : R(G) \longrightarrow K^0(X_n) \longrightarrow K^*(X_n)$

$\rightsquigarrow$ limit $\alpha : R(G) \longrightarrow \varprojlim K^*(X_n) =: K^*(BG)$.

In a 1961 paper, Atiyah shows:

(1) The AHSS can handle $\varprojlim$ in good cases, yielding
$$H^*(BG, \mathbb{Z}) \Rightarrow K^*(BG).$$

algebraic invariant

(2) The I-adic completion $\hat{\alpha} : \widehat{R(G)} \xrightarrow{\sim} K^*(BG)$ is a topological isomorphism.

topological invariant

Upshot: the AHSS is $E_2^* = H^*(G, \mathbb{Z}) \Rightarrow \widehat{R(G)}$, with $E_\infty^p = R(G)_p / R(G)_{p+1}$ for some filtration on $R(G)$.

Worked example: Let $G = C_n = \langle \sigma \mid \sigma^n = 1 \rangle$. Then we have a projective resolution of $\mathbb{Z}$, for $N = 1 + \sigma + \cdots + \sigma^{n-1}$,
$$\cdots \xrightarrow{\sigma - 1} \mathbb{Z}[G] \xrightarrow{N} \mathbb{Z}[G] \xrightarrow{\sigma - 1} \mathbb{Z}[G] \xrightarrow{\text{aug}} \mathbb{Z} \longrightarrow 0.$$
Applying $\mathrm{Hom}_{\mathbb{Z}[G]}(-, \mathbb{Z})$ and taking cohomology, we get
$$H^k(C_n, \mathbb{Z}) = \begin{cases} C_n, & k \text{ even} \geq 2, \\ 0, & k \text{ odd} \geq 1. \end{cases} \qquad (*)$$

On the other hand, if $\chi : C_n \to \mathbb{C}^\times$, $\sigma \mapsto e^{2\pi i/n}$, then

$$R(C_n) = \mathbb{Z}[\chi] \cong \mathbb{Z}[C_n]$$

with $I = (y)$ for $y = \chi - 1$. Note that

$$0 = \chi^n - 1 = (1+y)^n - 1 \equiv ny \pmod{y^2},$$

So in general $ny^k \equiv 0 \pmod{y^{k+1}}$ and $I^k/I^{k+1} \cong C_n$.
The filtration on $R(G)$ is then $R(G)_{2k-1} = R(G)_{2k} = I^k$, $k \geq 1$.

Since (*) is concentrated in even degrees, Atiyah's construction gives

$$H^*(C_n, \mathbb{Z}) = \mathrm{Gr}\, R(G) = \mathbb{Z}[y]/ny.$$

$$K^*(BG) = \widehat{R}(G) = \mathbb{Z}[[y]]/((y+1)^n - 1)$$

This can be checked topologically for $BC_2 = \mathbb{RP}^\infty = \bigcup \mathbb{RP}^n$, or more generally BC_n = infinite lens spaces.

Remark: Similar statements hold for compact connected Lie groups, proved jointly by AH. A general Atiyah–Segal completion theorem for equivariant K-theory with respect to compact Lie groups was established in 1969.

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