Spectral theory and approximation of Koopman operators in chaos

Part 2: Banach spaces that give you quasicompactness

Caroline Wormell

The University of Sydney

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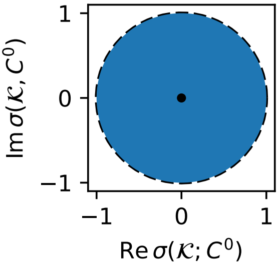
Recap

We looked at the contraction $f(x) = \kappa x$ on the domain $[-1, 1]$, and found that the spectrum depended on the function space:

$$\sigma(\mathcal{K}, C^r) = B(0, \kappa^r) \cup \{1, \kappa, \kappa^2, \ldots\}$$



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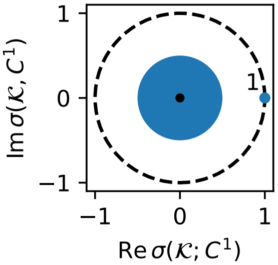

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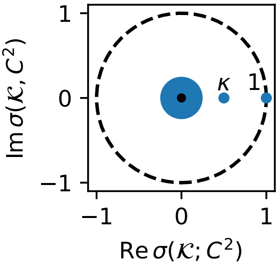

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Can we totally get rid of the essential spectrum? For our contraction, yes we can. It is not always true.

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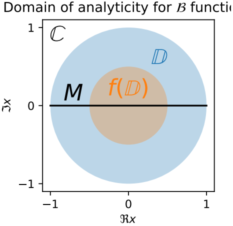
Let's go back to our original power series ansatz

$$\mathcal{B} = \{\psi(x) = \sum_{j=0}^{\infty} a_j x^j : \sum_{j=0}^{\infty} |a_j|^2 < \infty\} = \|(a_j)_j\|_{\ell^2}$$

with norm

$$\|\psi\|_{\mathcal{B}} = \sqrt{\sum_{j=0}^{\infty} |a_j|^2 < \infty}.$$

This is a really strong norm—it implies that ψ's Taylor series at 0 converges on a ball of radius 1.



Contraction in analytic function space

$$\|\psi\|_{\mathcal{B}} = \sqrt{\sum_{j=0}^{\infty} |a_j|^2}.$$

$$\mathcal{K} \sum_{j=0}^{\infty} a_j x^j = \sum_{j=0}^{\infty} \kappa^j a_j x^j$$

Nevertheless, we have accidentally created a reasonable-looking Hilbert space where our operator is self-adjoint (VERY RARE, WEIRD), and diagonal in our orthonormal basis $\{1, x, x^2, \ldots\}$:

$$\mathcal{K} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & \kappa & 0 & 0 & \cdots \\ 0 & 0 & \kappa^2 & 0 & \cdots \\ 0 & 0 & 0 & \kappa^3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

So we can just read off our spectrum: $\{1, \kappa, \kappa^2, \ldots\}$

Dynamical determinants

Aside: in fact, there is a function-space independent way to obtain these eigenvalues.

- You can take the trace of (some) infinite-dimensional operators. If $\mathcal{B}$ is a Hilbert space with orthonormal basis $\omega_0, \omega_1, \ldots$, this is given by

$$\operatorname{tr} \mathcal{K}^n = \sum_{b \in \mathbb{N}} \langle \omega_b, \mathcal{K} \omega_b \rangle_{\mathcal{B}}.$$

- Doing this for the Koopman operator, we typically find we get something like

$$\operatorname{tr} \mathcal{K}^n = \sum_{\substack{p : f^n(p) = p \\ Df^n(p) \text{ hyperbolic}}} \frac{1}{|\det(Df^n(p) - I)|}.$$

(Exercise: check this calculation for $f : \mathbb{R}/2\pi\mathbb{Z} \circlearrowleft$, using a Fourier basis to compute the trace.)

This looks like a formula that could work for any map.

Dynamical determinants

Using $\det(I + A) = \exp \operatorname{tr} \log(I + A)$, we can compute the *Fredholm determinant* of the resolvent for λ^{-1} small:

$$\zeta(\lambda^{-1}) := \det(I - \lambda^{-1}\mathcal{K}) = \exp \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \operatorname{tr} \mathcal{K}^n \lambda^{-n}$$

- ζ is known as a *dynamical zeta function*.
- It is analytic, so we can hopefully continue it to smaller λ^{-1}.
- Zeros with $|\lambda| > \rho_{\mathrm{ess}}(\mathcal{K}, \mathcal{B})$ should be discrete eigenvalues of $\mathcal{K}$ on $\mathcal{B}$, for reasonable $\mathcal{B}$.
- We call the (function space-independent) set of zeros *Ruelle–Pollicott resonances*.

Dynamical determinant example

For any contraction $f(x)$ on the interval with fixed point x_0 and $f'(x_0) = \kappa < 1$, we have

$$\operatorname{tr}\mathcal{K}^n = \frac{1}{|(f^n)'(x_0) - 1|} = \frac{1}{1 - \kappa^n} = \sum_{m=0}^{\infty} \kappa^{nm}.$$

(Compare with our contraction in basis x^n.)

$$\begin{aligned}\det(I - \mathcal{K}/\lambda) &= \exp \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \tfrac{(-1)^n}{n} (\kappa^m/\lambda)^n \\ &= \exp \sum_{m=1}^{\infty} \log(1 - \kappa^m/\lambda) \\ &= \prod_{m=1}^{\infty} (1 - \kappa^m/\lambda)\end{aligned}$$

which has zeroes at $1, \kappa, \kappa^2, \ldots$: our resonances!

Quasi-compactness recipe

How to get quasi-compactness in general?

Theorem (Lasota–Yorke, a.k.a. Doeblin–Fortet, a.k.a. Ionescu–Tulcea and Marinescu)

Given two Banach spaces $\mathcal{B}_s \subset \mathcal{B}_w$ (with different norms), if for all $v \in \mathcal{B}_w$ and $n \in \mathbb{N}$,

- *$\mathcal{B}_s$ is a compact, dense subset of $\mathcal{B}_w$;*
- *$\|\mathcal{A}^n v\|_{\mathcal{B}_w} \leq C\|v\|_{\mathcal{B}_w}$ for all $n > 0$;*
- *$\|\mathcal{A}^n v\|_{\mathcal{B}_s} \leq C_1 m^n \|v\|_{\mathcal{B}_s} + C_2 \|v\|_{\mathcal{B}_w}$ for all $n > 0$.*

Then $\rho(\mathcal{A}; \mathcal{B}_s) \leq 1$, and $\rho_{ess}(\mathcal{A}; \mathcal{B}_s) \leq m$.

- Note if $m = 0$ we recover compactness!
- This doesn't tell us much about the discrete spectrum—only that it can exist.

Intuition for quasi-compactness

- Generally, compactness implies some kind of regularising going on (e.g. the stochastic examples).
- *Intuition*: quasicompact = regularising also, but there's some infinite series that only contracts uniformly.
- Typical infinite things:
  - Arbitrarily high frequencies are allowed (i.e. C^r spaces)
  - The system generates an infinite sequence of singularities (Lecture 2)

$$\|\mathcal{K}^n v\|_{\mathcal{B}_s} \leq \underbrace{C_1 m^n \|v\|_{\mathcal{B}_s}}_{\substack{\text{CONTRACTING PART} \\ \text{action on high frequencies} \\ \text{dynamical complexity}}} + \underbrace{C_2 \|v\|_{\mathcal{B}_w}}_{\substack{\text{COMPACT PART} \\ \text{action on low frequencies} \\ \text{nonlinearities in the system}}}$$

General contractions

What if f is a general contraction (let's say C^1 for $r \geq 1$)? Let's suppose $|f'(x)| \leq \kappa$. Let's try $\mathcal{B}_s = C^1, \mathcal{B}_w = C^0$.

- C^1 is a compact, dense subset of C^0 (from yesterday—bounded functions and C^0 have the same norm)
- $\|\mathcal{K}^n\|_{C^0} = 1$
- We have

$$\begin{aligned}\|\mathcal{K}^n\psi\|_{C^1} &= \sup_x \left|(\psi \circ f^n)'(x)\right| + \|\mathcal{K}\psi\|_{C^0} \\ &= \sup_x \left|(f^n)'(x)\psi'(f^n(x))\right| + \|\mathcal{K}\psi\|_{C^0} \\ &\leq \sup_x \underbrace{\left|(f^n)'(x)\right|}_{\leq \kappa^n} \|\psi\|_{C^1} + \|\psi\|_{C^0}\end{aligned}$$

So $\rho(f, C^1) = 1$, $\rho_{\mathrm{ess}}(f, C^1) = |f'(x_0)|$.

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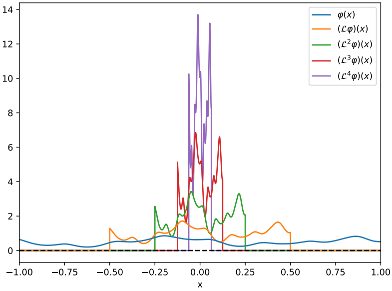
&\leq \sup_x \underbrace{\left|(f^n)'(x)\right|}_{\leq C|f'(x_0)|^n} \|\psi\|_{C^1} + \|\psi\|_{C^0}
\end{aligned}$$

So $\rho(f, C^1) = 1$, $\rho_{\mathrm{ess}}(f, C^1) = |f'(x_0)|$.

What about the transfer operator?

The transfer operator $\mathcal{L}$ tracks probability mass pushed by the dynamics.
For a contraction, this involves pushing mass together.



So we don't expect the transfer operator to be bounded in C^r.

Transfer operator of a contraction

But we know that $\mathcal{L}$ is kind of dual to $\mathcal{K}$, although it is not exactly $\mathcal{K}^*$. The difference is between functions and functionals:

$$(\mathcal{K}^*\chi)[\psi] = \chi[\mathcal{K}\psi]$$
$$\int (\mathcal{L}\varphi)(x)\,\psi(x)\,\mathrm{d}x = \int \varphi(x)\,(\mathcal{K}\psi)(x)\,\mathrm{d}x$$

So if $\mathcal{K}$ is nice on $\mathcal{B}$, we sort of want to define $\mathcal{L}$ to act on the dual space $\mathcal{B}^*$.

Dual space

The dual space of a Banach space $\mathcal{B}$ is, officially, the following

$$\mathcal{B}^* = \{\chi : \mathcal{B} \to \mathbb{R} : \|\chi\|_{\mathcal{B}\to\mathbb{R}} < \infty\},$$

with the following norm

$$\|\chi\|_{\mathcal{B}^*} = \|\chi\|_{\mathcal{B}\to\mathbb{R}} = \sup_{\|\psi\|_{\mathcal{B}}=1} |\chi(\psi)|$$

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For the purposes of this minicourse let's identify a function $\varphi : D \to \mathbb{R}$ with the functional

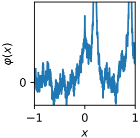
$$\psi \mapsto \int_D \varphi(x)\,\psi(x)\,\mathrm{d}x.$$

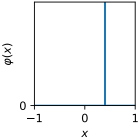
Transfer operator of a contraction

$\delta(x)$ obeys for all cts ψ
$\int \psi(x)\delta(x)\,\mathrm{d}x := \psi(0)$

So, ideally, we'd say that $\mathcal{L}$ acts $\mathcal{B}^* \to \mathcal{B}^*$. But $\mathcal{B}^*$ may not contain functionals that can be identified with functions...
For example, $(C^1)^*$ contains the following functionals (and hence the following "functions" φ)

- For any L^1 function φ, $\psi \mapsto \int_{-1}^{1} \varphi\,\psi\, x$
- For any $y \in (-1, 1)$:
  - $\psi \mapsto \psi(y)$ (i.e. $\varphi(x) = \delta(x-y)$)
  - $\psi \mapsto \psi'(y)$ (i.e. $\varphi(x) = -\delta'(x-y)$)
- For any L^1 function ϕ, $\psi \mapsto \int_{-1}^{1} -\phi\,\psi'\, x$
  (i.e.
  $\varphi(x) = \phi'(x) - \phi(-1)\delta(x+1) + \phi(1)\delta(x-1)$)
- For any measure μ supported on $[-1, 1]$:
  - $\psi \mapsto \int \psi\,\mathrm{d}\mu$ (i.e. $\varphi = \frac{\mathrm{d}\mu}{\mathrm{d}x}$)
  - $\psi \mapsto \int \psi'\,\mathrm{d}\mu$ (i.e. $\varphi = \frac{\mathrm{d}}{\mathrm{d}x}(\frac{\mathrm{d}\mu}{\mathrm{d}x})$)





Transfer operators in funny function spaces

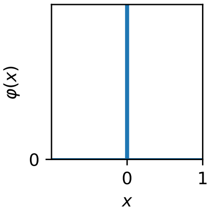
- This is fine! Most reasonable functions we'd want to try (e.g. C^∞ functions) live in $(C^1)^*$.
- We may get some extra stuff in eigenfunctions. Hopefully we can figure out how to interpret it.

Transfer operators in dual spaces

For our contraction, we'll get the same spectrum in $(C^r)^*$ as in C^r.
Just the eigenfunctions are different.
For the 1-eigenfunction, we have

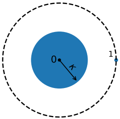
$$(\mathcal{L}\delta)(x) = \kappa^{-1}\delta(\kappa x) \underbrace{=}_{!} \delta(x)$$

(Exercise: check this black magic is correct by integrating against C^1 functions.)
This will give us an invariant measure $\mu = \delta$.



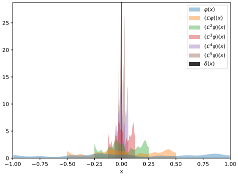
Transfer operators in dual spaces

In particular, in $(C^1)^*$ we get the following picture:



So we know that for any "hyperdistribution" $\varphi \in (C^1)^*$ (e.g. measure) with $\int \varphi \, \mathrm{d}x = 1$,

$$\|\mathcal{L}^n \varphi - \delta\|_{(C^1)^*} \leq C\kappa^n \|\varphi\|_{(C^1)^*}$$



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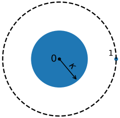

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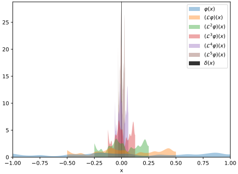
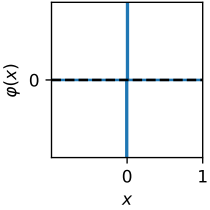
$$\underbrace{\|\mathcal{L}^n \varphi - \delta\|_{(C^1)^*}}_{\substack{= L^1 \text{ Wasserstein} \\ \text{distance between} \\ f_*^n \varphi \text{ and } \delta}} \leq C \kappa^n \|\varphi\|_{(C^1)^*}$$



Transfer operators in funny function spaces

For the κ-eigenfunction, we can differentiate:

$$(\mathcal{L}\delta')(x) = \kappa^{-1} \cdot \kappa\delta'(\kappa x) = \kappa\delta'(x)$$



In fact, the κ^n eigenfunction is the nth derivative of $\delta(x)$.

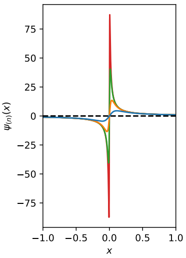
Transfer operators in dual spaces

Here we're saying that the transfer operator acts on functionals $\chi : C^r \to \mathbb{R}$, and then identifying some of these functionals with functions, and then extending the set of functions somehow...
How to do this in a less fluffy way?

Transfer operators in dual spaces

One quite robust way of defining these "functions" is as a completion of a nice space of functions on which your Koopman operator etc is just bounded (e.g. C^∞).

- *First*, you define a norm $\|\cdot\|$.
- Then, you complete C^∞ (etc.) with respect to the norm. That is, you define functions as limit points of $\|\cdot\|$-Cauchy sequences $\{\psi_{(n)}\}$ (up to equivalence).



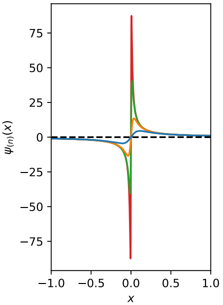
$$\psi := \lim_{n\to\infty} \psi_{(n)}$$

$$\frac{1+\sqrt{5}}{2} := \left\{\left(1, 2, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \ldots\right), \left(1, \frac{16}{10}, \frac{161}{100}, \frac{1618}{1000}, \ldots\right) \ldots\right\}$$

Transfer operators in dual spaces

This feels a bit abstract, but makes sense when we come to approximation:

$$\begin{array}{c}\text{eigenfunction of}\\ N\text{th approximation}\end{array} \xrightarrow[N\to\infty]{\mathcal{B}} \text{true eigen``function''}$$



Transfer operators in dual spaces

Nicely, it turns out that constructing Banach spaces from norms gives consistent eignefunctions:

Proposition

Suppose $\mathcal{B}_1, \mathcal{B}_2$ are the completion of $C^r(D)$ with respect to the norms $\|\cdot\|_1, \|\cdot\|_2$, and integrating against C^r functions is bounded in $\mathcal{B}_1, \mathcal{B}_2$.
Then if f is a C^{r+1} diffeomorphism, and $\mathcal{K}$ (resp. $\mathcal{L}$) is bounded on both $\mathcal{B}_1$ and $\mathcal{B}_2$, then their discrete spectra are consistent.

Weighted operators

There are various reasons you might want to study weighted operators:

$$(h\mathcal{K})\psi(x) = h(x)\mathcal{K}(f(x)) \text{ (default: } h = 1)$$
$$\mathcal{L}_g\varphi(x) = \sum_{f(y)=x} g(y)\varphi(y) \text{ (default: } g = 1/|\det Df|)$$

- We saw last lecture that escape rates from a set E depended on $\rho(\mathbb{1}_E\mathcal{K})$.
- You can get estimates on large deviations (e.g. Zhang *et al.* '22 in EDMD):

$$\mathbb{P}\left(\frac{1}{T}\sum_{t=0}^{T}\varphi(x_t) > c\right) \sim e^{-T\sup_{s\in\mathbb{R}}(cs-\log\rho(e^{s\varphi}\mathcal{K}))}$$

- You can compute Hausdorff dimensions of attractors. For example, in 2D, $d_s = \dim_H(\omega(f) - 1$ solves something like

$$\rho((|Df|_{E^s})^{-d_s}(\det Df)\mathcal{K}) = 1$$

Weighted operators

Lasota–Yorke is just an abstract functional analysis result, so we can apply it to weighted operators too:

Theorem (Lasota–Yorke and friends)

Given two Banach spaces $\mathcal{B}_s \subset \mathcal{B}_w$, if for all $v \in \mathcal{B}_w$ and $n \in \mathbb{N}$,

- *$\mathcal{B}_s$ is a compact, dense subset of $\mathcal{B}_w$;*
- $\|\mathcal{A}^n v\|_{\mathcal{B}_w} \leq C M^n \|v\|_{\mathcal{B}_w}$ *for all $n > 0$;*
- $\|\mathcal{A}^n v\|_{\mathcal{B}_s} \leq C_1 m^n \|v\|_{\mathcal{B}_s} + C_2 M^n \|v\|_{\mathcal{B}_w}$ *for all $n > 0$.*

Then $\rho(\mathcal{A}; \mathcal{B}_s) \leq M$, and $\rho_{ess}(\mathcal{A}; \mathcal{B}_s) \leq m$.

If the weight is non-negative, we in particular often have an eigenvalue at $\lambda = \rho(h\mathcal{K}) \in (0, M]$.

Operators in chaos

- We've looked a lot at contractions (=all negative Lyapunov exponents).
- What is the best way to study chaos (=some positive Lyapunov exponents on a compact set)?

Obviously we want to deal with something nice, like C^1...

Transfer operators in chaos

Let's look at the form of the transfer operator:

$$\mathcal{L}\varphi(x) = \sum_{f(y)=x} \frac{1}{|\det Df(y)|}\varphi(y)$$

Suppose that f is invertible. Then

$$\mathcal{L}\varphi(x) = \frac{1}{|\det Df(f^{-1}(x))|}\varphi(f^{-1}(x))$$

so it's a weighted Koopman operator of the inverse of f!

Transfer operators in chaos

Positive Lyapunov exponent means that at most points, you are expanding on average, in some directions.
Expansion is the inverse of contraction, so maybe what we know about contractions could help us?

Transfer operators in chaos

Let's start simple, with some uniformly expanding maps:
A map $f : [0,1] \to [0,1]$ is uniformly expanding if

- f is C^1 at all but a countable number of points
- When defined, $|f'(x)| \geq \gamma > 1$.

Such a map cannot be invertible! (Try integrating $|f'|$ over the domain.)

Transfer operators in chaos

Nomenclature:

- The domains of C^1-ness are called "branches", indexed by $i \in \mathcal{I}$
- On branch i, $f : \mathcal{O}_i \to \mathcal{N}_i$ (open sets) is a bijection, with inverse v_i.

The transfer operator of f is then:

$$\mathcal{L}\varphi(x) = \sum_{f(y)=x} \frac{1}{|f'(y)|}\varphi(y) = \sum_{i\in\mathcal{I}} \mathbb{1}_{\mathcal{N}_i}(x)\, \frac{1}{|f'(v_i(x))|}\varphi(v_i(x))$$

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Now by uniform expansion, we know

$$|v_i'(x)| = 1/|f'(v_i(x))| \leq \gamma^{-1} < 1$$

so our v_i are contractions...

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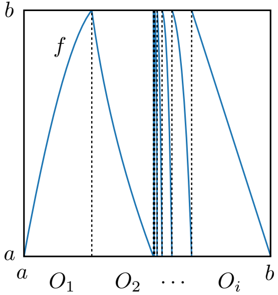
Transfer operators in chaos

Let's restrict to *full-branch* maps: those where the $\mathcal{N}_j$ form the whole interval:



Then

$$\mathcal{L}\varphi(x) = \sum_{i \in \mathcal{I}} \underbrace{|v_i'(x)|}_{=\sigma_i v_i'(x)} \varphi(v_i(x))$$

If the v_i are C^r, then we have a sum of weighted contraction operators on C^r, etc.

Lasota-Yorke inequality in chaos

Really L^1-type spaces are the nicest thing to deal with with transfer operators (because acting on measures), but with some extra work we can show that for $r > 0$,

$$\|\mathcal{L}\varphi\|_{C^{r-1}} \leq C\|\varphi\|_{C^{r-1}}$$
$$\|\mathcal{L}\varphi\|_{C^r} \leq Cm^n\|\varphi\|_{C^r} + C\|\varphi\|_{C^{r-1}}$$

where

$$m = \lim_{n\to\infty}\left(\sum_{\mathbf{i}\in I^n}\|v_{\mathbf{i}}'\|_{C^0}^n\right)^{1/n} = \rho(\mathcal{L}_{1/|f'|^{1+r}}) \leq \gamma^{-r}$$

So $\rho(\mathcal{L}, C^r) \leq 1$, and $\rho_{\mathrm{ess}}(\mathcal{L}, C^r) \leq \gamma^{-r}$.

Lasota-Yorke inequality in chaos

We can look at different kinds of spaces:

$$\|\psi\|_{W^{s,p}} = \|\psi\|_{L^p} + \|D^s\psi\|_{L^p}$$

where D^s is the (possibly non-integer) sth derivative.
In general, the integrability affects ρ_{ess} a bit, and differentiability away from 0 makes ρ_{ess} smaller:

Theorem (à la Baladi '18)

For smooth 1D full-branch expanding maps,

$$\rho_{ess}(\mathcal{L}; W^{s,p}) \leq q^{-1} \underbrace{\rho(\mathcal{L}_{(f')^{-1-qt}}; C^\alpha)}_{\text{space-independent}} \leq \gamma^{-t}$$

where $1/q + 1/p = 1$.

Different dynamics, different Banach spaces

We've seen that the Banach spaces you need to get meaningful spectra depend on the dynamics. For Koopman operators, we've seen

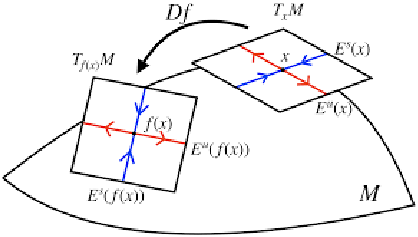
- Differentiable functions for contractions
- Hyper-distributions for uniformly expanding maps

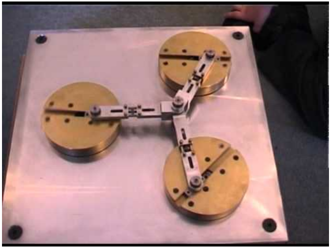
In general, we hope such Banach spaces exist, but...

Transfer operators in chaos

Chaos is difficult to prove things about. Here are some cool facts:

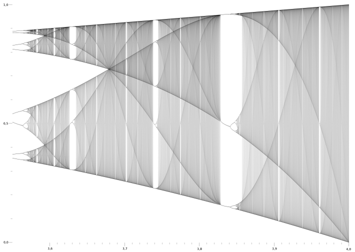
- There are a class of chaotic systems called "uniformly hyperbolic" aka "Anosov". We know some to all things about these systems. Almost no real dynamical systems are uniformly hyperbolic.





Transfer operators in chaos

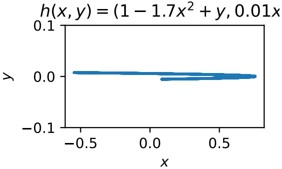
- Outside this class, our knowledge falls off very very fast[1]. The key sticking point is showing that systems even have chaotic attractors.
  - For example, give me some $a \in [3,4]$. In most cases can't confirm if the logistic map $f(x) = ax(1-x)$ has a chaotic attractor—because you have to know the whole future orbit of $x = 0$.



[1] Outside of some non-generic examples, e.g. the point of transition to chaos

Transfer operators in chaos

- In the 1990s M. Benedicks, L. Carlson, L.S. Young proved some $1 + \epsilon$-dimensional extensions of logistic maps have positive Lyapunov exponents.



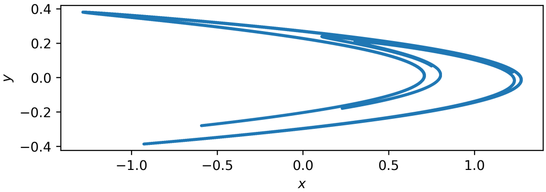
  No-one went further than that, because it was too hard.

- If you do have positive Lyapunov exponents, you can show a lot of mixing properties, but no explicit expression for the Banach spaces

Nevertheless, uniformly hyperbolic systems provide a good starting point for thinking about chaotic dynamics.

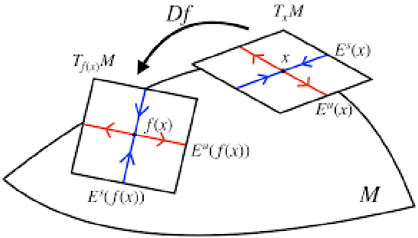
Anosov diffeomorphisms

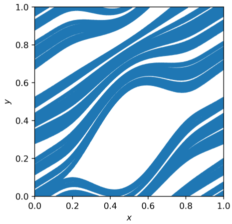
Here is a whirlwind tour of some other types of dynamics. Let's start with diffeomorphisms.
Invertibility means you must have contracting directions to balance the expansion.



Anosov diffeomorphisms

For Anosov diffeomorphisms there is a kind of local product structure in dynamics:





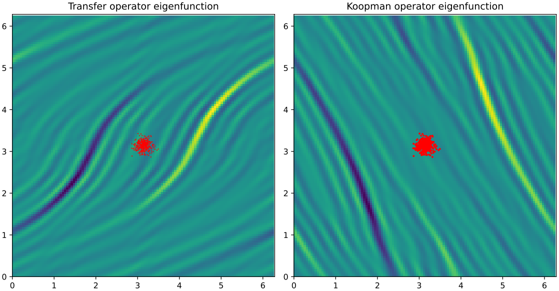
(Actually, this is true generally if you throw out some "bad" sets of small measure.)

Anosov diffeomorphisms

The way we construct a Banach space for the transfer operator is to construct a norm encoding

- Differentiability in unstable directions;
- Roughness in stable directions.

Vice versa for Koopman.



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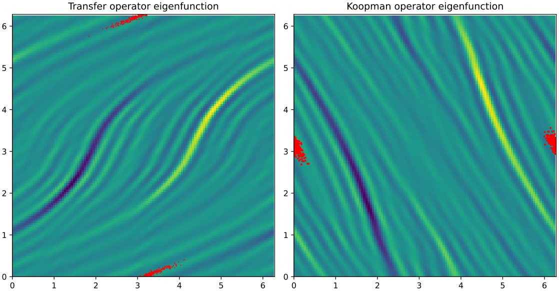

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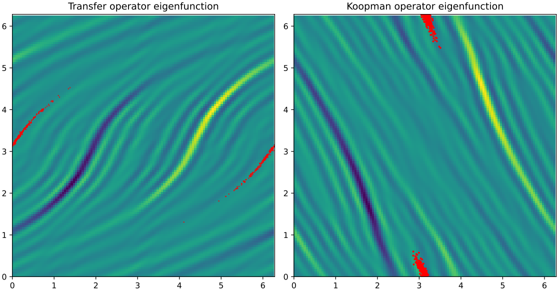

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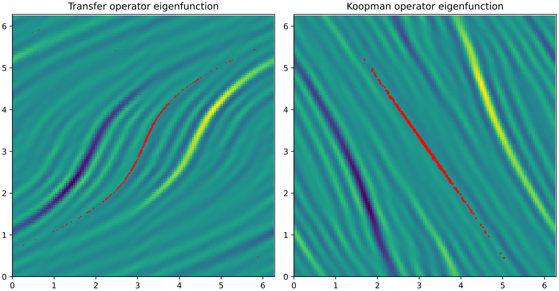

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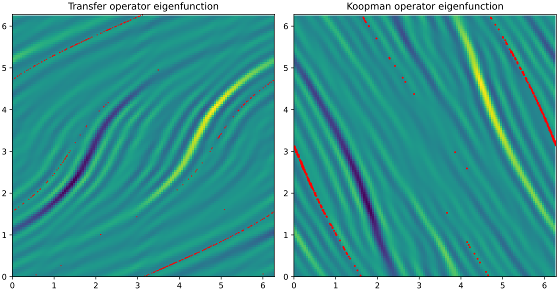


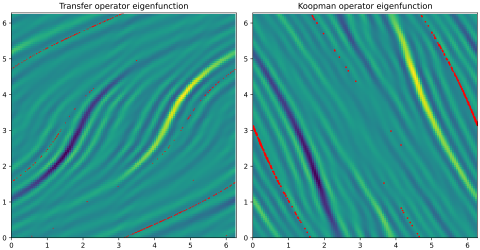
Anosov diffeomorphisms

The norms of these "anisotropic" spaces are fairly complicated even to define. There are maybe two main types:

- "Geometric" Banach spaces (Liverani, Blank, Demers, Gouëzel...), where you integrate over the rough directions:

$$\|\varphi\|_{p,s,t} = \text{"} \sup_{\substack{\alpha \approx \text{stable curve} \\ \|\psi\|_{C^t}=1}} \int_\alpha \nabla(\varphi \circ \alpha) \cdot \psi \, \mathrm{d}V \text{"}$$

None of these are Hilbert spaces.



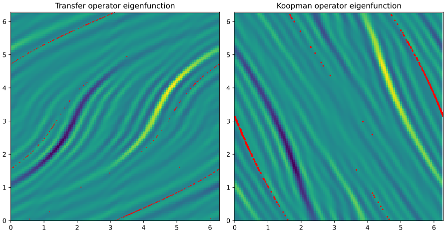
Anosov diffeomorphisms

- "Sobolev" spaces (Tsujii, Baladi,...), where you differentiate in various directions:

$$\|\varphi\|_{p,s,t} = \text{"}\|(\mathcal{F}^{-1} w_{s,t} \mathcal{F})\varphi\|_{L^p}\text{"}$$

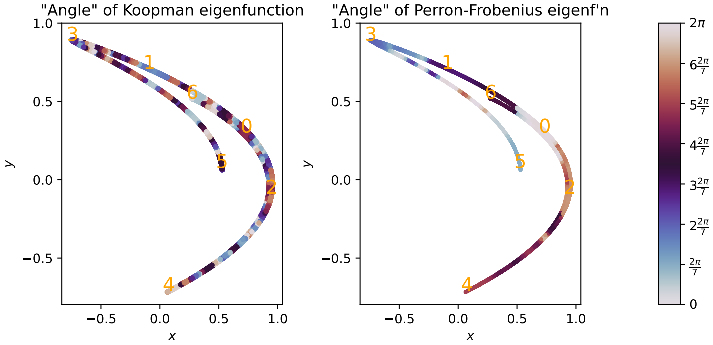
  - Triebel spaces:

$$\|\varphi\|_{p,s,t} = \text{"}\|(\mathrm{id} + \Delta)^s (\mathrm{id} + \Delta_u)^{-t/2} \varphi\|_{L^p}\text{"}$$



Non-uniformly hyperbolic maps

This kind of regularity structure persists in more realistic systems:



However, most of these systems systems *don't* have a product structure everywhere. This has some effects. . .