ON REGULARITY OF FINITE REFLECTION GROUPS

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Abstract. We define a concept of "regularity" for finite unitary reflection groups, and show that an irreducible finite unitary reflection group of rank greater than 1 is regular if and only if it is a Coxeter group. Hence we get a characterization of Coxeter groups among all the irreducible finite reflection groups of rank greater than one.

The irreducible finite unitary reflection groups were classified by Shephard and Todd ([12]) and by Cohen ([5]). They have been studied extensively by many people since then (see [2; 3; 4; 6; 7; 8; 9; 10; 11; 13]). Finite Coxeter groups are a special family of reflection groups, whose properties are relatively well known. It is interesting to ask what properties of Coxeter groups are shared also by the other reflection groups, and what are not. In the present paper we consider a property, which we call *regularity*, of a reflection group, defined by the existence of a *basic section* of an associated root system R (see 3.1). We show that an irreducible finite reflection group G of rank greater than 1 is regular if and only if all the root line circles in R are perfect (see 2.3 for the definitions). Then we further show that this holds if and only if G is a Coxeter group. Thus we get a characterization of Coxeter groups among all the finite reflection groups.

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Let G be an irreducible finite reflection group which acts irreducibly on a unitary space V, and let S be a simple reflection set for G (see 1.5). The strategy for proving our main result is as follows. It suffices to show that if G is complex and has rank greater than 1 then it is not regular. We show in Lemma 1.7 that the order $|Z(G)|$ of the center $Z(G)$ of G divides the cardinality of any root line in a root system R of G (see 1.6). We also show in Theorem 3.4 that G is regular if and only if all root line circles in R (see 2.1) are perfect. Suppose that G is regular and of rank greater than 1. Then we further show in Lemma 4.2 (1) that for any root α in R, the cardinality of the root line R_α is equal to the order of some $s \in S$. We show in Lemma 4.2 (2) that $|Z(G)|$ divides the orders $o(s)$ of all $s \in S$. We also show in Lemma 4.2 (3) that if the cardinality of each root line of R is 2, then G is a Coxeter group. By the classification of the irreducible finite unitary reflection groups, there are only three complex groups of rank greater than 1 satisfying both conditions (1), (2) of Lemma 4.2, namely, the groups G_8, G_{12} and G_{24} (in the notation of Shephard and Todd [12]). But the groups G_{12} and G_{24} are not regular by Lemma 4.2 (3), and we show that G_8 is not regular by exhibiting an an imperfect root line circle in its root system. This proves our main result.

The contents of the paper are organized as follows. In Section 1 we collect some definitions and results concerning irreducible finite reflection groups G which are either well known or easily proved. Then in Section 2 we introduce the concept of perfectness for a root line circle, a root system and a reflection group. Lemma 2.6, concerned with perfectness, is crucial in the proof of our main result. Regularity is introduced in Section 3, where we establish the equivalence of regularity of G and perfectness of its associated root system R (see Theorem 3.4). Our main result, Theorem 4.4, is proved in Section 4.

§1. Roots and reflections.

We collect some definitions and results concerning irreducible finite reflection groups; many of them follow from Cohen's paper [5].

1.1. Let V be a complex vector space of dimension n. A *reflection* on V is a linear transformation on V of finite order with exactly $n-1$ eigenvalues equal to 1. A *reflection group* G on V is a finite group generated by reflections on V. The group G is *reducible* if it is a direct product of two proper reflection subgroups and *irreducible* otherwise. The action of G on V is said to be *irreducible* if V has no nonzero proper G-invariant subspaces. In the present paper we shall always assume that G is irreducible and acts irreducibly on V. Call the dimension of V the *rank* of G. A reflection group G on V is called a *real* group or a *Coxeter group* if there is a G-invariant $\mathbb{R}$-subspace V_0 of V such that the canonical map $\mathbb{C} \otimes_{\mathbb{R}} V_0 \to V$ is bijective. If this is not the case, G will be called *complex.* (Note that, according to this definition, a real reflection group is not complex.)

Since G is finite, there exists a unitary inner product (,) on V invariant under G. From now on we assume that such an inner product is fixed.

1.2. A *root* of a reflection on V is an eigenvector corresponding to the unique nontrivial eigenvalue of the reflection. A *root of* G is a root of a reflection in G.

Let s be a reflection on V of order $d > 1$. There is a vector $a \in V$ of length 1 and a primitive d-th root ζ of unity such that $s = s_{a,\zeta}$, where $s_{a,\zeta}$ is defined by

$$s_{a,\zeta}(v) = v + (\zeta - 1)(v, a)a \tag{1.2.1}$$

for all $v \in V$. We also write $s_{a,d}$ for $s_{a,\zeta}$ if $\zeta = e^{2\pi i/d}$. Note that a can be chosen to be any root of s of length 1, and ζ is the nontrivial eigenvalue of s.

We use the notation $|x|$ for the cardinality of x if x is a set, and for the absolute value of x if x is a complex number. The meaning will always be clear from the context.

For each $v \in V$ define $o_G(v)$ to be the order of the (necessarily cyclic) group that consists of the identity and the reflections in G which have v as a root. (This group is $G_W = \{\, g \in G \mid gu = u$ for all $u \in W\,\}$, where $W = v^{\perp}$.) Thus $o_G(v) > 1$ if and only if v is a root of G. If a is a root of G, then $o_G(a)$ will be called the *order* of a (with respect to G). We shall denote $o_G(a)$ simply by $o(a)$ when G is clear from the context.

Note that we shall also use the notation $o(\zeta)$ for the order of ζ, where ζ could be either a root of unity, or a group element. This should cause no confusion.

Lemma 1.3. *We have $o(gv) = o(v) = o(cv)$ for all $v \in V$, $g \in G$ and $c \in \mathbb{C}^*$, where $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$.*

Proof. Let $W = v^\perp$. Then

$$u \in W \Longleftrightarrow (u, v) = 0 \Longleftrightarrow (gu, gv) = 0 \Longleftrightarrow gu \in (gv)^\perp.$$

This implies that $gW = (gv)^\perp$. Now we have

$$\begin{aligned} h \in G_{gW} &\Longleftrightarrow h(gu) = gu \quad \text{ for all } u \in W, \\ &\Longleftrightarrow (g^{-1}hg)u = u \quad \text{ for all } u \in W, \\ &\Longleftrightarrow g^{-1}hg \in G_W, \\ &\Longleftrightarrow h \in {}^gG_W. \end{aligned}$$

So we get $G_{gW} = {}^gG_W$ and hence $o(gv) = |G_{gW}| = |{}^gG_W| = |G_W| = o(v)$. The equation $o(v) = o(cv)$ follows from the fact that $v^\perp = (cv)^\perp$. $\square$

1.4. A pair (R, f) is called *a root system* in V, if

(i) R is a finite set of vectors of V of length 1;

(ii) $f : R \to \mathbb{N} \setminus \{1\}$ is a map such that $s_{a,f(a)}R = R$ and $f(s_{a,f(a)}(b)) = f(b)$ for all a, $b \in R$;

(iii) the group G generated by $\{\, s_{a,f(a)} \mid a \in R \,\}$ is a (finite) reflection group, and for all $a \in R$ and $c \in \mathbb{C}$,

$$ca \in R \Longleftrightarrow ca \in Ga.$$

The group G is called the reflection group associated with the root system (R, f). We have $o_G(a) = f(a)$ for any $a \in R$.

We shall denote a root system (R, f) simply by R when f is clear from the context.

1.5. *A system of simple roots* is a pair (B, w), where B is a finite set of vectors of V and w is a map from B to $\mathbb{N} \setminus \{1\}$, satisfying the following conditions:

(i) for all $a, b \in B$, we have $|(a,b)| = 1 \iff a = b$;

(ii) the group G generated by $S = \{\, s_{a,w(a)} \mid a \in B \,\}$ is finite;

(iii) there is a root system (R, f) with $R = GB$ and $f(a) = w(a)$ for all $a \in B$;

(iv) the group G cannot be generated by fewer than $|B|$ reflections.

We call the elements of S *simple reflections*. We also call (R, f) the root system of G *generated by* B, and B a *simple system* for R.

Note that we do not require B to be linearly independent. If B is linearly independent, then condition (iv) holds automatically.

The above definition of simple system is considerably weaker than the usual definition for Coxeter groups; in particular, it is not true that if B_1 and B_2 are simple systems for the same root system R then there is an element $g \in G$ with $gB_1 = B_2$.

It is easily seen that G acts irreducibly on V if and only if the root system R (resp. the simple root system B) spans V and cannot be decomposed into a disjoint union of two proper subsets R_1 and R_2 (resp. B_1 and B_2) with $R_1 \perp R_2$ (resp. $B_1 \perp B_2$).

By Lemma 1.3 we see that if $\alpha \in B$ and $\beta \in \mathbb{C}\alpha \cap R$, then $B' = (B \setminus \{\alpha\}) \cup \{\beta\}$ also forms a simple root system for R.

1.6. Let R be a root system in V with G the associated reflection group. For any $\alpha \in R$, let $R_\alpha = \mathbb{C}\alpha \cap R$, called *a root line* in R. By Property (iii) of a root system, we see that a root line of R is contained in a single G-orbit, and that the action of G on R induces a permutation action on $\overline{R}$, the set of all root lines of R. It is easily seen that if α, $\beta \in R$ then $s_\beta(R_\alpha) \neq R_\alpha$ unless $\beta \in R_\alpha$ or $(\beta, \alpha) = 0$.

We have the following result concerning the root lines.

Lemma 1.7. *Let R be a root system with G the associated reflection group. Then the order of the center $Z(G)$ of G divides $|R_\alpha|$ for any $\alpha \in R$.*

Proof. Since G acts irreducibly on V, it follows by Schur's Lemma that elements of $Z(G)$ act as scalar multipliers. Hence R_α is a union of some $Z(G)$-orbits, each of which has the same cardinality $|Z(G)|$. So the result follows. $\square$

1.8. In the subsequent sections, we always assume that G is an irreducible finite reflection group of rank greater than 1. Let R be an associated root system and B a simple system for R. Let $S = \{ s_\alpha \mid \alpha \in B \}$ be the set of the simple reflections associated with the simple system B; that is

$$s_\alpha(v) = v + (e^{2\pi i/d_\alpha} - 1)(v, \alpha)\alpha \qquad (\text{for all } v \in V)$$

where d_α is the order of α (and also of s_α).

1.9. Let (R, f) be a root system with G the associated reflection group. Then $s^k_{a,f(a)}$ is a reflection in G for any $a \in R$ and any integer k with $1 \leqslant k < f(a)$. Conversely, every reflection in G has such a form. For a, $b \in R$, if $1 \leqslant h < f(a)$ and $1 \leqslant k < f(b)$ then the reflections $s^h_{a,f(a)}$ and $s^k_{b,f(b)}$ are G-conjugate if and only if a and b are in the same G-orbit and $h = k$. It is possible that there is another root system (R', f') in V with the same associated reflection group G. In that case, we can find, for any $a \in R$, a root $a' \in R'$ such that $s_{a,f(a)} = s_{a',f'(a')}$. This implies that $a' \in \mathbb{C}a$. Similarly, for any $a' \in R'$, we can find some $a \in R$ with $a \in \mathbb{C}a'$. According to the condition (iii) of a root system, we see that the cardinality of a root line R_a in R is equal to that of the root line $R_{a'}$ in R'. In particular, when R consists of a single G-orbit, (R, f) is determined by G up to a scalar factor. These facts will be used in Section 4 in the proof of our main result.

§2. Perfectness.

2.1. Let $\overline{R}$ be the set of all root lines. A sequence $\xi : l_1, l_2, \ldots, l_r$ in $\overline{R}$ is called a *circle* if $r > 1$ and the following conditions hold.

(i) $l_i \neq l_{i+1}$ for all $i = 1, 2, \ldots, r$, with the convention that $l_{r+1} = l_1$.

(ii) For every h with $1 \leqslant h \leqslant r$, there exists some $s_h \in S$ satisfying $s_h(l_h) = l_{h+1}$.

2.2. Given two distinct root lines l and l' such that $l' = s(l)$ for some $s \in S$, it is possible that there exists another $s' \in S$ with $s'(l) = l'$. (This happens, for example, for the group G_7 of Shephard and Todd). Thus the sequence of simple reflections $s_1, s_2, \ldots, s_r$ associated with a root line circle ξ need not be unique.

2.3. We say that a root line circle $\xi : l_1, l_2, \ldots, l_r$ in $\overline{R}$ is *perfect* (with respect to S), if for any sequence of simple reflections $s_1, \ldots, s_r$ satisfying $s_h(l_h) = l_{h+1}$ for all h, we have $s_r s_{r-1} \cdots s_1(\alpha) = \alpha$ for each $\alpha \in l_1$. Otherwise we say that the root line circle is *imperfect*.

Remark 2.4. (1) If one root α in l_1 satisfies the condition that $s_r s_{r-1} \cdots s_1(\alpha) = \alpha$, then so do all the others. So when checking perfectness we need only consider a fixed $\alpha \in l_1$. However, we do need to consider every possible sequence of simple reflections $s_1, s_2, \ldots, s_r$ associated with the root line circle, since, as explained in 2.2 above, it need not be unique.

(2) Perfectness is defined relative to a set S of simple reflections for G. In the sequel, S is always fixed, and so we shall not mention the dependence on S explicitly.

2.5. Let $R = \bigcup_{i \in I} R_i$ be the decomposition of R into a disjoint union of G-orbits R_i, and $\overline{R} = \bigcup_{i \in I} \overline{R}_i$ the corresponding decomposition of $\overline{R}$. We say that R and $\overline{R}$ (resp. R_i and $\overline{R}_i$) are *perfect* if all the root line circles in $\overline{R}$ (resp. $\overline{R}_i$) are perfect (relative to S). We say that G is *perfect* if its associated root system R is perfect.

Lemma 2.6. *Let $\overline{R}_i$ be a perfect G-orbit in $\overline{R}$. Suppose that we have two root line sequences $\xi : l_1, \ldots, l_t$ and $\eta : l'_1, \ldots, l'_r$ in $\overline{R}_i$ with $l_1 = l'_1$ and $l_t = l'_r$, and two simple reflection sequences $s_1, \ldots, s_{t-1}$ and $s'_1, \ldots, s'_{r-1}$ such that $s_h(l_h) = l_{h+1} \neq l_h$ and $s'_k(l'_k) = l'_{k+1} \neq l'_k$ for all h and k (with $1 \leqslant h < t$ and $1 \leqslant k < r$). Take any root $\alpha \in l_1$. Then the equation*

$$s_{t-1} s_{t-2} \cdots s_2 s_1(\alpha) = s'_{r-1} s'_{r-2} \cdots s'_2 s'_1(\alpha)$$

is satisfied.

Proof. Let $\delta = s_{t-1} s_{t-2} \cdots s_2 s_1(\alpha)$ and $\delta' = s'_{r-1} s'_{r-2} \cdots s'_2 s'_1(\alpha)$, and suppose that the result is false. Then $\delta' = c\delta$ for some complex number $c \neq 1$. For all h and k with $1 \leqslant h \leqslant t$ and $1 \leqslant k \leqslant r$, put $\alpha_h = s_{h-1} s_{h-2} \cdots s_1(\alpha)$ and $\beta_k = s'_{k-1} s'_{k-2} \cdots s'_1(\alpha)$. Thus $\alpha_t = \delta$ and $\beta_r = \delta'$, and $\alpha_1 = \beta_1 = \alpha$.

Define a new sequence of roots $\zeta' : \mu_1 = \delta, \mu_2, \ldots, \mu_p = \alpha$ from the sequence $\zeta : \alpha_1 = \alpha, \alpha_2, \ldots, \alpha_t = \delta$ as follows. Start by reversing the order of ζ to get the sequence $\zeta^- : \alpha_t, \ldots, \alpha_2, \alpha_1$; then, for each h ($1 \leqslant h < t$) such that the order m_h of s_h is greater than 2, insert the sequence of roots

$$s_h(\alpha_{h+1}),\ s_h^2(\alpha_{h+1}),\ \ldots,\ s_h^{m_h-2}(\alpha_{h+1})$$

between α_{h+1} and α_h in ζ^-. This gives the desired sequence ζ'. Now we define τ to be the root line sequence $l_1, l_2, \ldots, l_{p+r-1}$ such that l_h contains μ_h for $1 \leqslant h \leqslant p$ and l_{p+k} contains β_k for $1 \leqslant k < r$. Then τ is an imperfect circle, contradicting our assumption. $\square$

§3. Basic sections of root systems.

As before, let B be a simple system for a root system R, let G be the associated reflection group and $S = \{ s_\alpha \mid \alpha \in B \}$ the set of simple reflections.

3.1. A subset $R' \subset R$ is called a *section* of R if it contains a unique root in each root line $l \in \overline{R}$. A section R' of R is *basic* if, for every $\beta \in B$, the simple reflection s_β permutes the roots in the set $R' \setminus R_\beta$. Equivalently, the section R' is basic if $s(\gamma) \in R'$ whenever $\gamma \in R'$ and s is a simple reflection such that $s(R_\gamma) \neq R_\gamma$.

Basicness, like perfectness, is defined relative to the set S, which we shall consider to be fixed.

A positive root system of a Coxeter group forms a basic section of its root system. Also, any root of a reflection group of rank one forms a basic section.

3.2. In general, a root system always contains a section, but need not contain a basic section. A finite reflection group G is called *regular* if there exists a simple system B for a root system R of G such that R contains a basic section relative to the simple reflection set of G corresponding to B.

The main goal of the present paper is to show that an irreducible finite reflection group G is regular if and only if it is either a Coxeter group or of rank 1.

By Lemma 2.6, we see that if all the root line circles of R are perfect then we can define a section R' of R in the following way. Let $R = \bigcup_{i \in I} R_i$ be the decomposition of R into a disjoint union of G-orbits R_i, and $\overline{R} = \bigcup_{i \in I} \overline{R}_i$ the corresponding decomposition of the root line set $\overline{R}$. For any $i \in I$, fix a root α in R_i, and let $l = R_\alpha \in \overline{R}_i$ be the root line containing α. For any root line $l' \in \overline{R}_i$ there exists a sequence $\xi : l_1 = l,\ l_2,\ \dots,\ l_r = l'$ in $\overline{R}_i$ such that $l_h \neq l_{h+1}$ and $s_h(l_h) = l_{h+1}$ for $1 \leqslant h < r$ and some $s_h \in S$. Define a root $\gamma \in l'$ by $\gamma = s_{r-1}s_{r-2}\cdots s_1(\alpha)$. By Lemma 2.6, γ depends only on α and not on the choice of ξ; so each l' contains a unique such γ. Let $R^{(i)}$ be the set of all roots γ obtained from α in this way for $l' \in \overline{R}_i$. Then the set $R' = \bigcup_{i \in I} R^{(i)}$ is a section of R.

Lemma 3.3. *The section R' defined above is basic.*

Proof. Let $\beta \in B$ and $\gamma \in R' \setminus R_\beta$. It suffices to prove that $s_\beta(\gamma) \in R'$. We may assume that $(\beta, \gamma) \neq 0$, since otherwise $s_\beta(\gamma) = \gamma \in R'$.

Let R_i be the G-orbit of R in which γ lies, and let α be the fixed root in R_i used in defining R'. By the definition of R' there exist root lines $R_\alpha = l_1,\ l_2,\ \dots,\ l_r = R_\gamma$ and simple reflections $s_1,\ s_2,\ \dots,\ s_{r-1}$ such that $l_h \neq l_{h+1}$ and $s_h(l_h) = l_{h+1}$ for $1 \leqslant h < r$, and $s_{r-1}\cdots s_1(\alpha) = \gamma$. Now as $\beta \notin \mathbb{C}\gamma$ and $(\beta, \gamma) \neq 0$, it follows that $s_\beta(l_r) = s_\beta(R_\gamma) \neq R_\gamma$, and so if we define $l_{r+1} = s_\beta(l_r)$ then the conditions $l_h \neq l_{h+1}$ and $s_h(l_h) = l_{h+1}$ hold for $1 \leqslant h < r+1$. Hence $s_\beta(\gamma) = s_r s_{r-1}\cdots s_1(\alpha)$ is in R', as required. $\square$

Lemma 3.3 immediately yields the “if” part of the following Theorem.

Theorem 3.4. *An irreducible finite reflection group G is regular if and only if all the root line circles in $\overline{R}$ are perfect.*

Proof. By Lemma 3.3, it suffices to prove the “only if” part. Let R' be a basic section of R. Let $\xi : l_1,\ \dots,\ l_r$ be a root line circle of $\overline{R}$ and $s_1,\ \dots,\ s_r$ simple reflections with $s_h(l_h) = l_{h+1}$ for $1 \leqslant h \leqslant r$. Choose a root $\alpha_1 \in l_1$. By Remark 2.4 (1) we can assume without loss of generality that $\alpha_1 \in R'$. Proceeding inductively, define $\alpha_{j+1} = s_j(\alpha_j)$ for $1 \leqslant j \leqslant r$. Since $s_j(l_j) \neq l_j$ it follows from the definition of a basic section that $\alpha_j \in R'$ for all j. Since $\alpha_{r+1} \in l_1 \cap R'$ and since α_1 is the unique root in $l_1 \cap R'$, this implies that $\alpha_{r+1} = \alpha_1$, and so the root line circle ξ is perfect. $\square$

§4. The main result.

In this section, we assume that G is a reflection group of rank greater than 1 and that (R, o_G) is an associated root system.

Lemma 4.1. *Assume that G is regular. Then for any $\alpha \in R$, the root line R_α has cardinality $o_G(\alpha)$.*

Proof. According to 1.5, we can choose a simple system B and a basic section R' for R with $B \subset R'$. Without loss of generality, we may assume that $\alpha \in R'$. Since s_α acts on R_α by multiplication by the scalar $\zeta = e^{2\pi i/o(\alpha)}$, it follows that R_α is a union of $\langle s_\alpha \rangle$-orbits, each orbit having the same cardinality $o(\alpha)$. We shall show that R_α actually consists of a single $\langle s_\alpha \rangle$-orbit, and hence has cardinality $o(\alpha)$.

Let Z be the subgroup of $\mathbb{C}^*$ generated by ζ, and let $\gamma \in R_\alpha$ be arbitrary. Since $R_\alpha = \mathbb{C}\alpha \cap G\alpha$, there exists an element $y \in G$ with $\gamma = y(\alpha)$. We may choose $s_1, s_2, \ldots, s_r \in S$ such that $y = s_r s_{r-1} \cdots s_1$. Put $\alpha_0 = \alpha$, and for $1 \leqslant k \leqslant r$, put $\alpha_k = s_k s_{k-1} \cdots s_1(\alpha)$ and let l_k be the root line containing α_k. We use induction on k to show that for each k, there exists an $\alpha'_k \in R'$ such that $\alpha_k \in Z\alpha'_k$.

The case $k = 0$ is trivial since $\alpha_0 = \alpha \in R'$. So assume that $k > 1$ and that $\alpha_{k-1} = \zeta^m \alpha'_{k-1}$ for some $\alpha'_{k-1} \in R'$ and $m \in \mathbb{Z}$. If $s_k(l_{k-1}) \neq l_{k-1}$ then by the definition of a basic section the root $s_k(\alpha'_{k-1})$ is in R'; so defining $\alpha'_k = s_k(\alpha'_{k-1})$, we conclude that $\alpha_k = s_k(\alpha_{k-1}) = \zeta^m \alpha'_k$, as required. Now assume that $s_k(l_{k-1}) = l_{k-1}$. Then either $\beta \in l_{k-1}$ or else $(\beta, \alpha_{k-1}) = 0$, where β is the simple root such that $s_\beta = s_k$. In the latter case $\alpha_k = s_\beta(\alpha_{k-1}) = \alpha_{k-1}$, and defining $\alpha'_k = \alpha'_{k-1} \in R'$ gives $\alpha_k \in Z\alpha'_k$ as required. On the other hand, if $\beta \in l_{k-1}$, then by Lemma 1.3, we have $o(\beta) = o_G(\alpha_{k-1}) = o_G(\alpha)$, since α_{k-1} and α are in the same G-orbit. Hence s_β acts on l_{k-1} by multiplication by ζ; so $\alpha_k = s_\beta(\alpha_{k-1}) = \zeta\alpha_{k-1} = \zeta^{m+1}\alpha'_k$, where $\alpha'_k = \alpha'_{k-1} \in R'$, and this completes the induction.

Since $\gamma = \alpha_r$, we conclude from the above that $\gamma \in Z\alpha'_r$, where $\alpha'_r \in R' \cap l_r$. But $l_r = R_\gamma = R_\alpha$, and $\alpha \in R'$ by hypothesis. Since R' is a section, it follows that $\alpha'_r = \alpha$, and thus $\gamma \in Z\alpha$. But γ was chosen as an arbitrary element of R_α, and so we conclude that $R_\alpha = Z\alpha$, as required. $\square$

Lemma 4.2. *Assume that G is regular.*

(1) For any $\alpha \in R$, we have $|R_\alpha| = o(s)$ for some $s \in S$.

(2) $|Z(G)|$ is a divisor of $o(s)$ for every $s \in S$.

(3) If the cardinality of each root line of R is 2, then G is a Coxeter group.

Proof. Since $R = GB$, there is an element $\beta \in B$ with $\alpha = g(\beta)$ for some $g \in G$. By Lemmas 4.1 and 1.3, we have $|R_\alpha| = o_G(\alpha) = o_G(\beta) = o(s_\beta)$. So (1) follows. Then (2) follows from (1), Lemma 1.7 and $R = GB$. Finally, under the assumption of (3), we can choose a simple system B and a basic section R' for R with $B \subset R'$. Then $R = R' \cup (-R')$. Let $S = \{\, s_\alpha \mid \alpha \in B \,\}$ be the corresponding simple reflection set of G. For each $s = s_\alpha \in S$, define P_s to be the set of all w in G such that $w^{-1}(\alpha) \in R'$. Clearly, P_s contains the identity element of G, and $P_s \cap sP_s = \emptyset$ for $s \in S$. Since $(sw)^{-1}(\alpha) = -w^{-1}(\alpha)$, it is clear that $w \in P_s$ if and only if $sw \notin P_s$. If $w \in P_s$ and $ws_\beta \notin P_s$ for some $\beta \in B$, then $\gamma = w^{-1}(\alpha)$ is in R' while $s_\beta(\gamma) \notin R'$. This forces $\gamma = \beta$. That is, $w^{-1}(\alpha) = \beta$, which implies that $w^{-1}s_\alpha w = s_\beta$. We conclude that (G, S) is a Coxeter system by [1, Ch. IV, No. 1.7, Proposition 6]. $\square$

4.3. Let G be a reflection group irreducibly acting on a unitary space V with R an associated root system. Two simple systems $B = \{e_1, \cdots, e_r\}$ and $B' = \{e'_1, \cdots, e'_t\}$ for R are said to be *equivalent* if $\{\, R_{e'_j} \mid 1 \leqslant j \leqslant t \,\} = \{\, R_{ge_i} \mid 1 \leqslant i \leqslant r \,\}$ for some $g \in G$. (In particular, this implies that $r = t$.) Given two equivalent simple systems B and B' for a root system R, it is easily seen that R is perfect with respect to B if and only if it is perfect with respect to B'.

We now show that the group G_8 of the list of Shephard and Todd is not regular. Let V be a unitary space of dimension 2 and take an orthonormal basis ϵ_1, ϵ_2 in V. Then $e_1 = \epsilon_1$, $e_2 = \frac{1+i}{2}(\epsilon_2 - \epsilon_1)$ are unit vectors spanning V, where $i^2 = -1$. Define $R = \{\, c\gamma_i \mid c \in \{\pm 1, \pm i\},\ 1 \leqslant i \leqslant 6 \,\}$ where $\gamma_1 = e_1$, $\gamma_2 = e_2$, $\gamma_3 = e_1 - ie_2$, $\gamma_4 = e_1 + e_2$, $\gamma_5 = e_1 + (1-i)e_2$ and $\gamma_6 = (1+i)e_1 + e_2$. Then $(R, 4)$ is a root system of $G = G_8$.

Let $B = \{e_1, e_2\}$. Then $(B, 4)$ is the unique simple root system for R up to equivalence. We have

$$\gamma_2 \xmapsto{s_1} \gamma_4 \xmapsto{s_2} \gamma_1 \xmapsto{s_2} \gamma_3 \xmapsto{s_1} -i\gamma_2,$$

where $s_i = s_{e_i}$, and the notation $a \xmapsto{s} b$ means $s(a) = b$. Then R_{γ_2}, R_{γ_4}, R_{γ_1}, R_{γ_3} form an imperfect root line circle in $\overline{R}$. Thus $\overline{R}$ contains an imperfect root line circle for any choice of simple system, and so it follows from Theorem 3.4 that G_8 is not regular.

We are now ready to prove our main result.

Theorem 4.4. *An irreducible finite reflection group G is regular if and only if G is either a Coxeter group or of rank one.*

Proof. The implication $\Leftarrow$ is well known (see 3.1); so it suffices to show that if G is irreducible, complex and has rank greater than 1 then G is not regular. Assume, for a contradiction, that G is such a group and is regular. Then G must satisfy the conditions (1) and (2) of Lemma 4.2. According to the classification of the irreducible finite complex reflection groups, G can only be one of the groups G_8, G_{12}, G_{24} by [2; 5]. Since the cardinality of any root line in a root system of G_{12} or G_{24} is 2, this implies that none of these two groups is regular by Lemma 4.2 (3). Finally, G_8 is not regular, by 4.3 above. This proves our result. $\square$

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