Group representation theory **Lecture 19, 13/10/97**

Proposition. *Let H, K be subgroups of the finite group G, and let $\lambda\colon H \to \mathbb{C}^{\times}$ and $\mu\colon K \to \mathbb{C}^{\times}$ representations of degree 1. Let $e = (1/|H|)\sum_{h\in H}\lambda(h^{-1})h$ and $f = (1/|K|)\sum_{k\in K}\mu(k^{-1}k$. If there exists an $x \in H \cap K$ such that $\lambda(x) \neq \mu(x)$ then $ef = 0$.*

Proof. Assume that such an element x exists. Then

$$ex = \sum_{h\in H}\lambda(h^{-1})hx = \sum_{h\in H}\lambda(xl^{-1})l = \lambda(x)\sum_{h\in H}\lambda(l^{-1})l = \lambda(x)e.$$

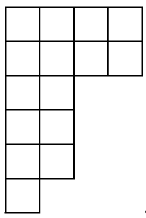
Similarly

$$xf = \sum_{k\in K}\mu(k^{-1})xk = \sum_{k\in K}\mu(l^{-1}x)l = \mu(x)\sum_{k\in K}\mu(l^{-1})l = \mu(x)f.$$

Hence $\lambda(x)ef = exf = \mu(x)ef$, and since $\lambda(x) \neq \mu(x)$ it follows that $ef = 0$. □

Representation theory of the symmetric group

A *partition* of a positive integer n is a finite nonincreasing sequence of positive integers whose sum is n. Thus, for example $(4,4,2,2,2,1)$ is a partition of 15. The *table* corresponding to a partition $(n_1, n_2, \ldots, n_k)$ of n is a sequence of k rows of boxes, with n_i boxes in the row i, arranged so that the jth box in row $i+1$ is placed directly below the jth box in row i. Thus the table corresponding to the above partition of 15 is



A *diagram* is obtained by filling the boxes of the table with the numbers from 1 to n (in any order). For example,

1	2	3	4
5	6	7	8
9	10		
11	12		
13	14		
15			

and

1	3	6	7
2	4	10	13
5	8		
9	12		
11	14		
15			

are two diagrams corresponding to this same partition. Of course there are precisely $n!$ diagrams for each partition of n.

Let D be a diagram corresponding to some partition of n. We shall say that numbers i and j are *collinear* in D if they appear in the same row of D, and *co-columnar* if they appear in the same column of D. The *row group* $R(D)$ of D is the set of all permutations σ of $\{1, 2, \ldots, n\}$ such that σi is in the same row of D as i, for each $i \in \{1, 2, \ldots, n\}$:

$$R(D) = \{\, \sigma \in S_n \mid \ i \text{ and } \sigma i \text{ are collinear in } D \text{ for each } i \,\}.$$

Similarly, the *column group* of D is

$$C(D) = \{\, \sigma \in S_n \mid \ i \text{ and } \sigma i \text{ are co-columnar in } D \text{ for each } i \,\}.$$

It is clear that $R(D)$ and $C(D)$ are subgroups of S_n. Thus if D is the second of the two examples above then the column group of D is isomorphic to the direct product $S_6 \times S_5 \times S_2 \times S_2$. Indeed,

$$\begin{aligned} C(D) &= \mathrm{Sym}\{1,2,5,9,11,15\} \times \mathrm{Sym}\{3,4,8,12,14\} \times \mathrm{Sym}\{6,10\} \times \mathrm{Sym}\{7,13\} \\ &= \bigl\{\, \sigma_1\sigma_2\sigma_3\sigma_4 \mid \sigma_1 \in \mathrm{Sym}\{1,2,5,9,11,15\},\ \sigma_2 \in \mathrm{Sym}\{3,4,8,12,14\}, \\ &\qquad\qquad\qquad\qquad\qquad\qquad \sigma_3 \in \mathrm{Sym}\{6,10\},\ \sigma_4 \in \mathrm{Sym}\{7,13\} \,\bigr\}. \end{aligned}$$

Similarly the row group of D is

$$R(D) = \mathrm{Sym}\{1,3,6,7\} \times \mathrm{Sym}\{2,4,10,13\} \times \mathrm{Sym}\{5,8\} \times \mathrm{Sym}\{9,12\} \times \mathrm{Sym}\{11,14\} \times \mathrm{Sym}\{15\},$$

where of course the last factor is a trivial group.

Our aim is to construct a collection of minimal left ideals in the group algebra $\mathbb{C}S_n$, and the following notation will be useful for this purpose. If H is any subgroup of S_n we define

$$\begin{aligned} [H]_1 &= \sum_{\sigma \in H} \sigma \\ [H]_\varepsilon &= \sum_{\sigma \in H} \varepsilon(\sigma)\sigma \end{aligned}$$

where $\varepsilon(\sigma)$ is 1 if σ is an even permutation, -1 if σ is odd. It will transpire that if D is any diagram then the element $e(D) = [R(D)]_1[C(D)]_\varepsilon$ is a scalar multiple of a primitive idempotent in $\mathbb{C}S_n$, so that $\mathbb{C}S_n e(D)$ is a minimal left ideal of $\mathbb{C}S_n$.

For example, let D be the diagram

1	2
3	

so that $R(D)$ is the group of order 2 generated by the transposition $(1,2)$, and $C(D)$ similarly has order 2 and is generated by $(1,3)$. Then

$$e(D) = (\mathrm{id} + (1,2))(\mathrm{id} - (1,3) = \mathrm{id} + (1,2) - (1,3) - (1,3,2).$$

The left ideal $\mathbb{C}S_3 e(D)$ is the linear space spanned by the elements $\sigma e(D)$ obtained as σ runs through all six elements of S_3. But since $(1,2)[R(D)]_1 = [R(D)]_1$ it follows that $\sigma e(D) = \sigma(1,2)e(D)$ for each value of σ, and so only three distinct products $\sigma e(D)$ are obtained as σ varies. These are

$$\begin{aligned} e(D) &= \mathrm{id} + (1,2) - (1,3) - (1,3,2) \\ (2,3)e(D) &= (2,3) + (1,3,2) - (1,2,3) - (1,2) \end{aligned}$$

and

$$(1,2,3)e(D) = (1,2,3) + (1,3) - (2,3) - \mathrm{id},$$

and since the sum of these three is zero we deduce that the left ideal generated by $e(D)$ is a two-dimensional left $\mathbb{C}S_3$ module. Taking $e(D)$ and $(2,3)e(D)$ as a basis we can easily compute the matrix $\phi(\sigma)$ of the linear transformation of this module given by left multiplying by any element $\sigma \in S_3$. For example,

$$\phi(1,2) = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \qquad \text{and} \qquad \phi(1,2,3) = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$$

and it follows readily that the character of the representation ϕ is the irreducible character of S_3 of degree 2.

Let π be a partition of n. The symmetric group S_n has a permutation action on the set of all diagrams corresponding to π, as follows: if D is a diagram and $\sigma \in S_n$ then σD is the diagram obtained from D by replacing i by σi (for all $i \in \{1, 2, \ldots, n\}$). Thus, for example, if

$D =$

1	2	3	4
5	6	7	
8			
9			

and

$D' =$

3	8	5	9
2	4	7	
1			
6			

then $D' = (1, 3, 5, 2, 8)(4, 9, 6)(7)D$. For any partition π of n the action of S_n on the set of all diagrams corresponding to π is transitive, and the stabilizer of an element is trivial. Hence if D is any fixed diagram then $\sigma \leftrightarrow \sigma D$ is a one to one correspondence between S_n and the set of all diagrams for π: this permutation representation is thus essentially the regular representation of S_n.

Lemma. *Let D be a diagram for some partition of n, and let $\sigma \in S_n$. Then $R(\sigma D) = \sigma R(D)\sigma^{-1}$, and $C(\sigma D) = \sigma C(D)\sigma^{-1}$.*

Proof. Let i, $j \in \{1, 2, \ldots, n\}$. Then i, j are collinear in D if and only if σi and σj are collinear in σD. Thus, for all $\tau \in S_n$, the following condition

$$i \text{ and } \tau i \text{ are collinear in } D \text{ for all } i \in \{1, 2, \ldots, n\} \tag{1}$$

is equivalent to

$$\sigma i \text{ and } \sigma(\tau i) \text{ are collinear in } \sigma D \text{ for all } i \in \{1, 2, \ldots, n\},$$

and if we put $j = \sigma i$ this becomes

$$j \text{ and } (\sigma\tau\sigma^{-1})j \text{ are collinear in } \sigma D \text{ for all } j \in \{1, 2, \ldots, n\}. \tag{2}$$

Now $\tau \in R(D)$ if and only if condition (1) holds, and $\sigma\tau\sigma^{-1} \in R(\sigma D)$ if and only if condition (2) holds. Since we have shown that (1) and (2) are equivalent it follows that

$$\sigma R(D)\sigma^{-1} = \{\, \sigma\tau\sigma^{-1} \mid \tau \in R(D) \,\} = \{\, \sigma\tau\sigma^{-1} \mid \sigma\tau\sigma^{-1} \in R(\sigma D) \,\} = R(\sigma D).$$

The proof that $C(\sigma D) = \sigma C(D)\sigma^{-1}$ is similar. □

Let $\pi = (n_1, n_2, \ldots, n_k)$ and $\pi' = (m_1, m_2, \ldots, m_l)$ be partitions of n. We say that $\pi > \pi'$ if there exists a j such that $n_j > m_j$ and $n_i = m_i$ for all $i < j$. This is the so-called *lexicographic ordering* of partitions: the first place in which two partitions differ determines which is greater. Clearly, if π, π' are distinct partitions then either $\pi > \pi'$ or $\pi' > \pi$; in other words, we have a total ordering of the set of all partitions of n.

The following combinatorial lemma is the key to our investigation of left ideals in $\mathbb{C}S_n$.

Lemma. *Let $(n_1, n_2, \ldots, n_k) \geq (m_1, m_2, \ldots, m_l)$ be of partitions of n, and let D, D' be diagrams for these partitions. Suppose that no two numbers are collinear in D and co-columnar in D'. Then the partitions are equal, and $D' = \rho\sigma D$ for some $\rho \in R(D)$ and $\sigma \in C(D)$.*

Proof. The n_1 numbers in the first row of D all lie in different columns of D'. But D' has n'_1 columns, and $n'_1 \leq n_1$. So $n'_1 = n_1$. Furthermore, each column of D' contains a unique number from the first row of D; so applying a suitable column permutation to D' will take these n_1 numbers into the first row. In other words, for some $\tau_1 \in C(D')$, the numbers in the first row of $\tau_1 D'$ are the same as the numbers in the first row of D (in some order). Note also that

$$C(\tau_1 D') = \tau_1 C(D')\tau_1^{-1} = C(D')$$

since $\tau_1 \in C(D')$. So numbers are co-columnar in $\tau_1 D'$ if and only if they are co-columnar in D'.

Note that since $n_1 = n'_1$ and $\pi \geq \pi'$ it follows that $n_2 \geq n'_2$. We now, in effect, cover up the first rows of our diagrams and repeat the argument on the remainder. The n_2 numbers in the second row of D all lie in different columns of $\tau_1 D'$ and not in the first row. But $\tau_1 D'$ has only $n'_2 \leq n_2$ columns which contain places outside the first row. So $n'_2 = n_2$, and each of these columns contains a unique number from the second row of D. Applying a suitable column permutation to $\tau_1 D'$ shifts these n_2 numbers to the second row without changing the first row. So we obtain a diagram $\tau_2\tau_1 D'$ which has the same numbers in the first row as D has in the first row, and also the same numbers in the second row as D has in the second row. Moreover, since $\tau_2 \in C(\tau_1 D') = C(D')$ it follows that the columns of $\tau_2\tau_1 D'$ are permutations of the corresponding columns of $\tau_1 D'$ and D', and we still have the property that no two numbers collinear in D are co-columnar in $\tau_2\tau_1 D'$.

Covering the first two rows and repeating the argument, and continuing on in this way, we find that $n'_i = n_i$ for all $i \in \{1, 2, \ldots, k\}$, and there exists a permutation $\tau = \tau_k\tau_{k-1}\cdots\tau_1 \in C(D')$ such that, for all i, the diagram $\tau D'$ has the same numbers in its ith row as D has. So there is a $\rho \in R(D)$ such that $\rho D = \tau D'$. Now since

$$\tau^{-1} \in C(\tau D') = C(\rho D) = \rho C(D)\rho^{-1},$$

it follows that if we put $\sigma = \rho^{-1}\tau^{-1}\rho$ then $\sigma \in C(D)$. Furthermore,

$$\rho\sigma D = (\rho\rho^{-1}\tau^{-1})(\rho D) = \tau^{-1}(\tau D') = D',$$

as required. □