Examples of metric spaces

(1) Let $S = \mathbb{C}^n = \{\,(x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{C}\,\}$, and let $p > 1$. For $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ in $\mathbb{C}^n$ define

$$d_p(x, y) = \Big(\sum_{k=1}^{n} |x_k - y_k|^p\Big)^{1/p}.$$

Then d_p is a metric.

To prove this one must check the axioms. First, since $|x_k - y_k| = |y_k - x_k| \geq 0$, it is obvious that $d_p(x, y) = d_p(y, x) \geq 0$ for all x and y. Furthermore, since $\sum_k |x_k - y_k|^p = 0$ if and only if all the terms $|x_k - y_k|^p$ are zero, we see that $d_p(x, y) = 0$ if and only if $x = y$. To verify the remaining axiom we use Minkowski's Inequality.

Let x, y, $z \in \mathbb{C}^n$, and define $a_k = y_k - x_k$ and $b_k = x_k - z_k$, where x_k, y_k and z_k are the k-th coordinates of x, y and z respectively. Then $a_k + b_k = y_k - z_k$, and so

$$\begin{aligned} d_p(y, z) &= \Big(\sum_{k=1}^{n} |a_k + b_k|^p\Big)^{1/p} \\ &\leq \Big(\sum_{k=1}^{n} |a_k|^p\Big)^{1/p} + \Big(\sum_{k=1}^{n} |b_k|^p\Big)^{1/p} \\ &= d_p(y, x) + d_p(z, x) \end{aligned}$$

as required.

Note that we could have used $\mathbb{R}^n$ rather than $\mathbb{C}^n$, and everything would have worked in the same way.

Two particular cases should be singled out. If $p = 2$ we get the usual Euclidean metric,

$$d(x, y) = \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2 + \cdots + |x_n - y_n|^2}.$$

If $p = 1$ we get

$$d(x, y) = |x_1 - y_1| + |x_2 - y_2| + \cdots + |x_n - y_n|,$$

which is the distance from x to y if you can only move parallel to the coordinate axes.

(2) Let S be any set. For x, $y \in S$, define

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y. \end{cases}$$

This is easily shown to be a metric; it is known as the *standard discrete metric* on S.

(3) Let d be the Euclidean metric on $\mathbb{R}^3$, and for x, $y \in \mathbb{R}^3$ define

$$d(x, y) = \begin{cases} d(x, y) & \text{if } x = sy \text{ or } y = sx \text{ for some } s \in \mathbb{R} \\ d(x, 0) + d(0, y) & \text{otherwise.} \end{cases}$$

This says that $d_1(x, y)$ is the distance from x to y if you can only travel along rays through the origin. Then d_1 is a metric.

(4) Let $S = \mathbb{C}^n$ and d_p as in the first example above. We considered the case $p = 1$; so it is natural to ask about the case $q = 1$, where p and q are related as in our discussion of Hölder's Inequality. The relationship is $1/p + 1/q = 1$, and if $q = 1$ this gives $1/p = 0$, which is nonsense of course. However, since $1/p \to 0$ as $p \to \infty$, perhaps it makes sense to define

$$d_\infty(x, y) = \lim_{p\to\infty} d_p(x, y)$$

and perhaps d_∞ will be a metric.

Let $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in \mathbb{C}_n$, and define

$$s = \max\{\, |x_1 - y_1|, |x_2 - y_2|, \ldots, |x_n - y_n| \,\}.$$

Then for all $p > 1$,

$$s^p \le |x_1 - y_1|^p + |x_2 - y_2|^p + \cdots + |x_n - y_n|^p \le ns^p,$$

and so taking p-th roots we obtain

$$s \le (|x_1 - y_1|^p + |x_2 - y_2|^p + \cdots + |x_n - y_n|^p)^{1/p} \le n^{1/p}s.$$

That is, $s \le d_p(x, y) \le n^{1/p}s$, for all $p > 1$. But $\lim_{p\to\infty} n^{1/p} = 1$; so by the squeeze law we deduce that $\lim_{p\to\infty} d_p(x, y) = s$.

Anyway, it is easy to check directly that d_∞ defined by $d_\infty(x, y) = \max_i |x_i - y_i|$ is a metric on $\mathbb{C}^n$.

More background

Recall that a subset X of $\mathbb{R}$ is said to be *bounded above* if there exists a $B \in \mathbb{R}$ such that $x \le B$ for all $x \in X$. It is an axiom of the real number system that every subset X of $\mathbb{R}$ which is bounded above has a least upper bound, or *supremum*, $\sup X$. Thus if X is bounded above the supremum of X has the following two properties:

(i) for all $x \in X$, $x \le \sup X$;

(ii) if $B \in \mathbb{R}$ satisfied $x \le B$ for all $x \in X$ then $\sup X \le B$.

More examples

Let $p \ge 1$, and define ℓ^p to be the set of all infinite sequences $(x_k)_{k=1}^\infty$ in $\mathbb{R}$ such that $\sum_{k=1}^\infty |x_k|^p$ converges. If $a = (a_k)_{k=1}^\infty$ and $b = (b_k)_{k=1}^\infty$ are any elements of ℓ^p, then for each positive integer n Minkowski's Inequality gives

$$\Bigl(\sum_{k=1}^n |a_k - b_k|^p\Bigr)^{1/p} \le \Bigl(\sum_{k=1}^n |a_k|^p\Bigr)^{1/p} + \Bigl(\sum_{k=1}^n |b_k|^p\Bigr)^{1/p} \le A^{1/p} + B^{1/p},$$

where $A = \sum_{k=1}^\infty |a_k|^p$ and $B = \sum_{k=1}^\infty |b_k|^p$. Thus $\sum_{k=1}^n |a_k - b_k|^p \le (A^{1/p} + B^{1/p})^p$ for all n, and so the series $\sum_{k=1}^\infty |a_k - b_k|^p$ is convergent. Defining

$$d(a, b) = \bigl(\sum_{k=1}^\infty |a_k - b_k|^p\bigr)^{1/p};$$

makes ℓ^p into a metric space. The only nontrivial thing to prove is the triangle inequality, and this is quite easy also: if (x_k), (y_k), $(z_k) \in \ell^p$ then by Minkowski's Inequality

$$\begin{aligned}\Big(\sum_{k=1}^{n} |y_k - z_k|^p\Big)^{1/p} &\leq \Big(\sum_{k=1}^{n} |y_k - x_k|^p\Big)^{1/p} + \Big(\sum_{k=1}^{n} |x_k - z_k|^p\Big)^{1/p} \\ &\leq \Big(\sum_{k=1}^{\infty} |x_k - y_k|^p\Big)^{1/p} + \Big(\sum_{k=1}^{\infty} |x_k - z_k|^p\Big)^{1/p} \\ &= d(x,y) + d(x,z)\end{aligned}$$

for all n, and taking the limit as $n \to \infty$ gives $d(y,z) \leq d(x,y) + d(x,z)$.

Let $[\alpha,\beta]$ be any finite closed interval in $\mathbb{R}$, and $\mathcal{C}[\alpha,\beta]$ the set of all continuous real-valued functions on $[\alpha,\beta]$. Let $p \geq 1$, and for f, $g \in \mathcal{C}[\alpha,\beta]$ define

$$d(f,g) = d_p(f,g) = \Big(\int_{\alpha}^{\beta} |f(x) - g(x)|^p \, dx\Big)^{1/p}.$$

With this definition of distance, $\mathcal{C}[\alpha,\beta]$ becomes a metric space.

Again, the proof of the triangle inequality uses Minkowski's Inequality. By the definition of the Riemann integral, if ϕ is any element of $\mathcal{C}[\alpha,\beta]$ then

$$\int_{\alpha}^{\beta} \phi(x)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} \phi(x_i)/n$$

where $x_i = \alpha + (i(\beta - \alpha)/n)$. Hence

$$\lim_{n\to\infty} \Big(\sum_{i=1}^{n} \phi(x_i)/n\Big)^{1/p} = \Big(\int_{\alpha}^{\beta} \phi(x)\, dx\Big)^{1/p}.$$

Now let f, g, h be arbitrary elements of $\mathcal{C}[\alpha,\beta]$, and let $n \in \mathbb{Z}^+$. By Minkowski's identity applied with $a_i = (g(x_i) - f(x_i))/n^{1/p}$ and $b_i = (f(x_i) - h(x_i))/n^{1/p}$,

$$\Big(\sum_{i=1}^{n} |g(x_i) - h(x_i)|^p/n\Big)^{1/p} \leq \Big(\sum_{i=1}^{n} |g(x_i) - f(x_i)|^p/n\Big)^{1/p} + \Big(\sum_{i=1}^{n} |f(x_i) - h(x_i)|^p/n\Big)^{1/p}.$$

Taking limits as $n \to \infty$, we deduce that

$$\begin{aligned}\Big(\int_{\alpha}^{\beta} |g(x) - h(x)|^p \, dx\Big)^{1/p} &= \lim_{n\to\infty} \Big(\sum_{i=1}^{n} |g(x_i) - h(x_i)|^p/n\Big)^{1/p} \\ &\leq \lim_{n\to\infty} \Big(\sum_{i=1}^{n} |f(x_i) - g(x_i)|^p/n\Big)^{1/p} + \lim_{n\to\infty} \Big(\sum_{i=1}^{n} |f(x_i) - h(x_i)|^p/n\Big)^{1/p} \\ &= \Big(\int_{\alpha}^{\beta} |f(x) - g(x)|^p \, dx\Big|\Big)^{1/p} + \Big(\int_{\alpha}^{\beta} |f(x) - h(x)|^p \, dx\Big|\Big)^{1/p},\end{aligned}$$

so that $d(g,h) \le d(f,g) + d(f,h)$, as required.

The metric d in this example is called the L^p metric on $\mathcal{C}[\alpha,\beta]$. In both this example and the previous one we can "put $p=\infty$", obtaining a metric on the set of bounded infinite sequences given by

$$d_\infty(a,b) = \sup_k |a_k - b_k|$$

and a metric on $\mathcal{C}[\alpha,\beta]$ given by

$$d_\infty(f,g) = \sup_{x\in[\alpha,\beta]} |f(x) - g(x)|.$$

It is easy to check by direct verification of the axioms that these are metrics.

The following proof was not done in lectures, and is added here just for interest's sake. We show that $\lim_{p\to\infty} d_p(f,g) = d_\infty(f,g)$ for all f, $g \in \mathcal{C}[\alpha,\beta]$, provided that $\alpha < \beta$.

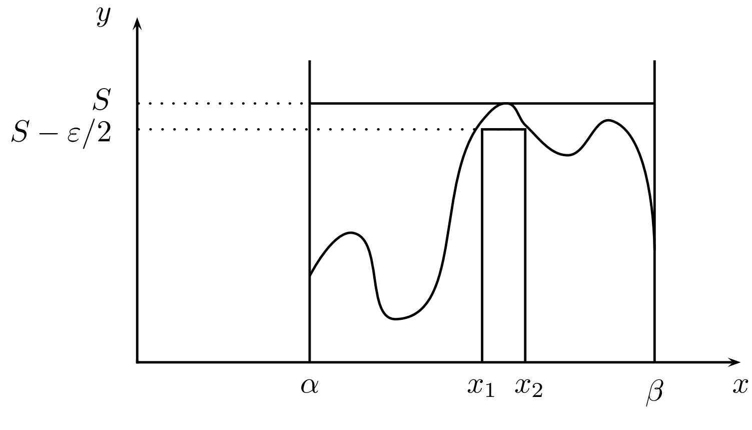
Let f, $g \in \mathcal{C}[\alpha,\beta]$ and let

$$S = d_\infty(f,g) = \sup_{x\in[\alpha,\beta]} |f(x) - g(x)|.$$

Let $\varepsilon > 0$ be arbitrary. There exists $x_0 \in [\alpha,\beta]$ such that $|f(x_0) - g(x_0)| > S - \frac{\varepsilon}{2}$, and by continuity of $|f(x) - g(x)|$ it follows that $|f(x) - g(x)| > S - \frac{\varepsilon}{2}$ for all x close enough to x_0. That is, there exist x_1, x_2 with $\alpha \le x_1 < x_2 \le \beta$ such that $|f(x) - g(x)| > S - \frac{\varepsilon}{2}$ for all $x \in [x_1, x_2]$. Define $\phi(x)$ on $[\alpha,\beta]$ by

$$\phi(x) = \begin{cases} S - \frac{\varepsilon}{2} & \text{if } x \in [x_1, x_2] \\ 0 & \text{otherwise,} \end{cases}$$

and observe that $\phi(x) \le |f(x) - g(x)| \le S$ for all $x \in [\alpha,\beta]$. A typical example of this situation is illustrated in the diagram below, in which the graphs of $\phi(x)$, $|f(x) - g(x)|$ and the constant function S on the interval $[\alpha,\beta]$ are shown.



For all $p \ge 1$ we have $\phi(x)^p \le |f(x) - g(x)|^p \le S^p$ for all $x \in [\alpha,\beta]$, and so

$$\begin{aligned}(x_2 - x_1)(S - \tfrac{\varepsilon}{2})^p &= \int_\alpha^\beta \phi(x)^p \, dx \\ &\le \int_\alpha^\beta |f(x) - g(x)|^p \, dx \\ &\le \int_\alpha^\beta S^p \, dx = (\beta - \alpha) S^p.\end{aligned}$$

Taking p-th roots gives

$$(x_2 - x_1)^{1/p}(S - \tfrac{\varepsilon}{2}) \le \Bigl(\int_\alpha^\beta |f(x) - g(x)|^p \, dx\Bigr)^{1/p} \le (\beta - \alpha)^{1/p} S,$$

and since the limit as $p \to \infty$ of the pth root of any positive number is 1, the left hand side approaches $S - \frac{\varepsilon}{2}$ and the right hand side approaches S as $p \to \infty$. So there exists $P \in \mathbb{R}$ such that $(x_2 - x_1)^{1/p}(S - \frac{\varepsilon}{2}) > S - \varepsilon$ and $(\beta - \alpha)^{1/p} S < S + \varepsilon$ for all $p > P$. Hence for all $p > P$,

$$S - \varepsilon < \Bigl(\int_\alpha^\beta |f(x) - g(x)|^p \, dx\Bigr)^{1/p} < S + \varepsilon,$$

and since ε was arbitrary, this shows that $\lim\limits_{p\to\infty} \Bigl(\int_\alpha^\beta |f(x) - g(x)|^p \, dx\Bigr)^{1/p} = S$, as claimed.

Let (X, d) be any metric space and let Y be an arbitrary subset of X. We can define a distance function $d_Y \colon Y \times Y \to \mathbb{R}$ by

$$d_Y(a, b) = d(a, b) \qquad \text{for all } a,\, b \in Y.$$

In other words, d_Y is the restriction to $Y \times Y$ of the distance function $d \colon X \times X \to \mathbb{R}$. It is clear that d_Y satisfies the distance function axioms, since d does; thus (Y, d_Y) is a metric space. We say that (Y, d_Y) is a *subspace* of (X, d), and d_Y is called the metric *induced* on Y by the metric d on X.

Topology

Let (X, d) be a metric space. Recall that if $A \subseteq X$ then a point $a \in A$ is called an interior point of A if there exists $\varepsilon > 0$ such that $B(a, \varepsilon) \subseteq A$. The set of all interior points of A is $\mathrm{Int}(A)$, the interior of A. It is clear that $\mathrm{Int}(A) \subseteq A$ for all subsets A of X.

Lemma. *Let A be an arbitrary subset of the metric space X. Then* $\mathrm{Int}(\mathrm{Int}(A)) = \mathrm{Int}(A)$.

Proof. The inclusion $\mathrm{Int}(\mathrm{Int}(A)) \subseteq \mathrm{Int}(A)$ is immediate from the comment preceding the statement of the lemma; so we just have to prove the reverse inclusion.

Let $a \in \mathrm{Int}(A)$ be arbitrary. Choose $\varepsilon > 0$ such that $B(a, \varepsilon) \subseteq A$. We shall show that in fact $B(a, \varepsilon) \subseteq \mathrm{Int}(A)$. For, suppose that $b \in B(a, \varepsilon)$. Since $B(a, \varepsilon)$ is open, there exists $\delta > 0$ such that $B(b, \delta) \subseteq B(a, \varepsilon)$. Thus $B(b, \delta) \subseteq A$, which shows that $b \in \mathrm{Int}(A)$. As this holds for all $b \in B(a, \varepsilon)$, we have shown that $B(a, \varepsilon) \subseteq \mathrm{Int}(A)$, as claimed. However, this statement says that a is an interior point of $\mathrm{Int}(A)$, and since a was originally chosen as an arbitrary point of $\mathrm{Int}(A)$, we have shown that all points of $\mathrm{Int}(A)$ are interior points, as required. $\square$