THE UNIVERSITY OF SYDNEY
PURE MATHEMATICS 3901

Metric Spaces 2000

Tutorial 11

1. Let (X,d) be a metric space. Prove that the following statements are equivalent:

(i) X is disconnected;
(ii) there exist two nonempty disjoint open subsets A_1, A_2 in X such that $X = A_1 \cup A_2$;
(iii) there exist two nonempty disjoint closed subsets A_1, A_2 in X such that $X = A_1 \cup A_2$;
(iv) there exist a nonempty proper subset of X which is both open and closed in X.

Solution.

Suppose that X is disconnected. So X is the union of two nonempty separated sets. That is, there exist nonempty subsets A_1, A_2 of X such that $X = A_1 \cup A_2$ and $A_1 \cap \overline{A_2} = \emptyset = \overline{A_1} \cap A_2$. Now $A_1 \cap \overline{A_2} = \emptyset$ implies that $A_1 \subseteq X \setminus \overline{A_2}$ (the complement of $\overline{A_2}$), while $X = A_1 \cup A_2$ and $A_2 \subseteq \overline{A_2}$ give $X \setminus \overline{A_2} \subseteq X \setminus A_2 \subseteq A_1 = A_1 \cup A_2 \setminus A_2 \subseteq A_1$. So $A_1 = X \setminus \overline{A_2}$, and so A_1 is open in X (as it is the complement of the closed set $\overline{A_2}$). Similarly $A_2 = X \setminus \overline{A_1}$ is open in X. And $A_1 \cap A_2 \subseteq A_1 \cap \overline{A_2} = \emptyset$. So A_1 and A_2 are nonempty disjoint open sets whose union is X. So we have shown that (i) implies (ii).

Now abandon the assumptions of the previous paragraph and assume instead that (ii) holds: that is, assume that A_1 and A_2 are open and nonempty, $X = A_1 \cup A_2$ and $A_1 \cap A_2 = \emptyset$. Then A_1 is the complement of the open set A_2, and so A_1 is closed. Similarly $A_2 = X \setminus A_1$ is closed. So X is the disjoint union of the nonempty closed sets A_1 and A_2, whence (iii) holds.

Assume that (iii) holds. Since $A_1 = X \setminus A_2$ and A_2 is closed, it follows that A_1 is open. Since X_2 is nonempty, $X \setminus A_2$ is a proper subset of X. Thus A_1 is a nonempty proper subset which is both open and closed in X. (Of course, so is A_2.) So (iv) holds.

Assume (iv). Let A_1 be a nonempty proper subset of X which is both open and closed in X, and let $A_2 = X \setminus A_1$, so that $X = A_1 \cup A_2$ and $A_1 \cap A_2 = \emptyset$. Since A_1 is open, its complement, A_2 is closed. So $\overline{A_2} = A_2$, and $\overline{A_1} = A_1$ (since A_1 is closed). So $\overline{A_1} \cap A_2 = A_1 \cap \overline{A_2} = A_1 \cap A_2 = \emptyset$. So A_1 and A_2 are separated, and since $X = A_1 \cup A_2$ we have shown that X is disconnected. So (i) holds.

Hence all the statements are equivalent.

2. Let A be a closed and bounded set in $\mathbb{R}$ and let $p = \sup A$. Prove that $p \in A$.

Solution.

Note that the assumption that A is bounded guarantees that A has a supremum. (Actually, we only need A to be bounded above.) Now suppose, for a contradiction, that $p \notin A$. Then $p \in \mathbb{R} \setminus A$, which is open since A is closed. So there exists an $\varepsilon > 0$ such that the open interval $(p-\varepsilon, p+\varepsilon) = B(p,\varepsilon)$ is contained in $X \setminus A$. Now let $a \in A$ be arbitrary. Then $a \le p$ since p is an upper bound for A. If $a > p - \varepsilon$ then $a \in (p-\varepsilon, p] \subseteq (p-\varepsilon, p+\varepsilon) \subseteq \mathbb{R} \setminus A$, contradicting $a \in A$. So $a \le p - \varepsilon$, and since this holds for all $a \in A$ it follows that $p - \varepsilon$ is an upper bound for A. Since p is the least upper bound for A it follows that $p \le p - \varepsilon$, contradicting $\varepsilon > 0$.

3. Provide two examples of disconnected sets A with $\bar{A}$ connected.

Solution.

Let $X = \mathbb{R}$, with the usual topology, and put $A = (0,1) \cup (1,2)$. Then A is not connected, but $\overline{A} = [0,2]$ is.

For another example, let $A \subseteq \mathbb{R}$ be the set of all irrational numbers. Then $A_1 = A \cap (0,\infty)$ and $A_2 = A \cap (-\infty, 0)$ are open subsets of A. (That is, they are open as subsets of A—open in the subspace topology on A.) Since $A = A_1 \cup A_2$ we see that A is disconnected. But $\overline{A} = \mathbb{R}$ is connected.

4. Let X be a metric space and $(A_i)_{i\in I}$ a family of connected subsets in X, and B a connected subset of X such that for each i, $A_i \cap B \ne \emptyset$. Prove that the union $A = B \cup \left(\bigcup_i A_i\right)$ is connected.

Solution.

Suppose, for a contradiction, that $A = U_1 \cup U_2$, where U_1 and U_2 are nonempty subsets of A that are open in the subspace topology on A. Since $B \subseteq A$ it follows that $B = (B \cap U_1) \cup (B \cap U_2)$, the union of two open sets of B. Since B is connected, either $B \cap U_1 = \emptyset$ and $B \cap U_2 = B$, or $B \cap U_2 = \emptyset$ and $B \cap U_1 = B$. Renumbering U_1 and U_2 if need be, we may assume the former alternative. So $B \cap U_2 = B$, which means that $B \subseteq U_2$.

Now for each $i \in I$ we have $A_i \subseteq A = U_1 \cup U_2$, and so $A_i = (A_1 \cap U_1) \cup (A_i \cap U_2)$, the union of two open sets of A_i. Since $\emptyset \ne A_i \cap B \subseteq A_i \cap U_2$, and since A_i

is connected, it follows that $A_i \cap U_1 = \emptyset$. This holds for all $i \in I$, and since also $B \cap U_1 = \emptyset$ it follows that

$$U_1 = A \cap U_1 = \big(B \cup \bigcup_i A_i\big) \cap U_1 = (B \cap U_1) \cup \bigcup_i (A_i \cap U_1) = \emptyset,$$

contradicting the original choice of U_1 and U_2.

5. Let $A \subseteq \mathbb{R}^n$ be a disconnected set with disconnection $U_1 \cup U_2$. Let B be a connected subset of A. Show that either $B \subset U_1$ or $B \subset U_2$.

Solution.

The same is true for any topological space X, we do not need to assume that we are dealing with $\mathbb{R}^n$.

Since $B \subseteq A$ and U_1 and U_2 are open in A, it follows that $B \cap U_1$ and $B \cap U_2$ are open in B. Since $B \subseteq U_1 \cup U_2$ it follows that $B = (B \cap U_1) \cup (B \cap U_2)$, and since B is connected one of the sets $B \cap U_i$ is empty and the other is B. But $B \cap U_i = B$ implies $B \subseteq U_i$; so either $B \subseteq U_1$ or $B \subseteq U_2$.

6. For each of the following pairs of sets X and Y, explain why they are not homeomorphic.

(i) $X = \mathbb{R},\ Y = (0, 2]$ *(ii)* $X = [0, 2),\ Y = [0, 2]$

(iii) $X = [0, 2],\ Y = \{\, (x_1, x_2) \in \mathbb{R}^2 \mid {x_1}^2 + {x_2}^2 = 4 \,\}$

Solution.

(i) Observe that $Y \setminus \{2\} = (0, 2)$ is a connected space. If there were a homeomorphism $f\colon Y \to X$ then $X \setminus \{f(2)\}$ would be homeomorphic to $Y \setminus \{2\}$, and hence connected. But $\mathbb{R} \setminus \{a\}$ is disconnected for all $a \in \mathbb{R}$; so no such homeomorphism exists.

(ii) Observe that $Y \setminus \{0, 2\} = (0, 2)$ is connected; so if X and Y were homeomorphic there would exist two (distinct) points a, $b \in X$ with $X \setminus \{a, b\}$ connected. Without loss of generality, suppose that $a < b$. Then since a, $b \in (0, 2]$ we have $0 < a < b \le 2$, and then $X \setminus \{a, b\}$ is either $(0, a) \cup (a, b) \cup (b, 2]$ (if $b \ne 2$) or $(0, a) \cup (a, 2)$ (if $b = 2$). In both cases it is disconnected.

(iii) Again, removal of 0 and 2 from X leaves a connected set, but the removal of any two distinct points from the circle Y will leave a disconnected set. So X and Y are not homeomorphic.

7. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function such that $(f(t))^2 = 4$ for all $t \in \mathbb{R}$. Prove that either $f(t) = 2$ for all $t \in \mathbb{R}$ or $f(t) = -2$ for all $t \in \mathbb{R}$.

Solution.

Let $U_1 = f^{-1}(2) = \{\, t \mid f(t) = 2 \,\}$, and $U_2 = f^{-1}(-2)$. Since $\{2\}$ and $\{-2\}$ are both closed in $\mathbb{R}$, and continuous preimages of closed sets are closed (by the definition of continuity), it follows that U_1 and U_2 are both closed. They are obviously disjoint, and since for all $t \in \mathbb{R}$ we have $(f(t))^2 = 4$, giving $f(t) = \pm 2$, either $t \in U_1$ or $t \in U_2$. So $\mathbb{R} = U_1 \cup U_2$. But $\mathbb{R}$ is connected, and so cannot be the disjoint union of two nonempty closed sets. So either $U_1 = \emptyset$ and $U_2 = \mathbb{R}$ or vice versa, whence either $f(t) = 2$ for all $t \in \mathbb{R}$ or $f(t) = -2$ for all $t \in \mathbb{R}$.

Supplementary questions

8. Use Question 4 and results from lectures to show that the unit square in $\mathbb{R}^2$, given by $S = \{\, (x, y) \in \mathbb{R}^2 \mid 0 \le x \le 1,\ 0 \le y \le 1 \,\}$, is a connected subset of $\mathbb{R}^2$.

Solution.

For each $y \in [0, 1]$ define $A_y = \{\, (x, y) \in \mathbb{R}^2 \mid 0 \le x \le 1\}$. It is clear that $x \mapsto (x, y)$ is a continuous map from $[0, 1]$ to A_y, and since $[0, 1]$ is connected (as proved in lectures) it follows that A_y is connected. Similarly, the set $B = \{\, (0, y) \in \mathbb{R}^2 \mid 0 \le y \le 1\}$ is connected. Moreover, $B \cap A_y \ne \emptyset$ for each y, since $(0, y) \in B \cap A_y$. By Question 4, $B \cup \bigcup_{y \in [0,1]} A_y$ is connected; that is, S is connected.

9. Use the previous exercise to show that $E = \{\, (x, y) \in \mathbb{R}^2 \mid 2x^2 + 3y^2 \le 1 \,\}$ is connected.

Solution.

The function $(x, y) \mapsto (\frac{1}{\sqrt{2}} x \cos(\pi y), \frac{1}{\sqrt{3}} x \sin(\pi y))$ is continuous since its component functions are differentiable, and it maps the square S onto the ellipse E. Since continuous images of connected sets are connected it follows that E is connected.