THE UNIVERSITY OF SYDNEY

PURE MATHEMATICS 3901

Metric Spaces 2000

Tutorial 1

1. Let X be a set and A, B and C subsets of X

(*i*) Prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

(*ii*) Prove that $X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B)$.

Solution.

(*i*) It is an elementary result of propositional logic that if p, q and r are any propositions then

$$(p \text{ or } (q \text{ and } r)) \quad \text{is equivalent to} \quad ((p \text{ or } q) \text{ and } (p \text{ or } r)). \qquad (*)$$

Hence the set $A \cup (B \cap C) = \{\, x \mid x \in A \text{ or } (x \in B \text{ and } x \in C \,\}$ coincides with $(A \cup B) \cap (A \cup C) = \{\, x \mid (x \in A \text{ or } x \in B) \text{ and } (x \in A \text{ or } x \in C) \,\}$.

The result $(*)$ (known as a *distributive law*) is commonly listed as one of the basic rules of inference of propositional logic: something that one may simply use without comment in proofs. The validity of the basic rules is established by means of truth tables. Thus for this particular distributive law one considers all eight possible combinations of truth values (true or false) for the propositions p, q and r, verifying that p or (q and r) is true exactly when (p or q) and (p or r) is.

It is also easy prove $(*)$ using other rules of inference which may possibly seem even more basic and obvious, such as

$$p \text{ implies } (p \text{ or } q)$$
$$(p \text{ and } q) \text{ implies } p$$

and such like. Thus, suppose that p or (q and r) is true. If p is false this means that q and r are both true, and therefore (p or q) and (p or r) are both true. On the other hand, if p is true then again it follows that (p or q) and (p or r) are both true. So (p or q) and (p or r) holds whenever p or (q and r) holds. Conversely, suppose that (p or q) and (p or r) both hold. If p is false then this means that q and r are both true; thus, either p is true or (q and r) is true, as required.

(*ii*) This is similar to Part (*i*): this time the relevant rule of propositional logic is that the negation of (p or q) is ((negation of p) and (negation of q)). Thus

$$\begin{aligned} X \setminus (A \cup B) &= \{\, x \in X \mid \text{not true that } (x \in A \text{ or } x \in B) \,\} \\ &= \{\, x \in X \mid (x \notin A) \text{ and } (x \notin B) \,\} \\ &= \{\, x \in X \mid x \notin A \,\} \cap \{\, x \in X \mid x \notin B \,\} \\ &= (X \setminus A) \cap (X \setminus B). \end{aligned}$$

2. For each positive integer i, let $A_k = (-1/k, 1/k)$ (an open interval in $\mathbb{R}$). Show that $\bigcap_{k=1}^{\infty} A_k = \{0\}$.

Solution.

For each $k \in \mathbb{Z}^+$ we have $-1/k < 0 < 1/k$; so $0 \in A_k$. Thus $0 \in \bigcap_{k=1}^{\infty} A_k$. It remains to show that $\bigcap_{k=1}^{\infty} A_k$ has no elements other than 0.

Suppose that $a \in \bigcap_{k=1}^{\infty} A_k$ and $a \neq 0$. Then $|a| > 0$, and since $\lim_{k\to\infty} 1/k = 0$ we may choose $K \in \mathbb{Z}^+$ such that $1/K < |a|$. Then $a \notin (-1/K,\ 1/K) = A_K$. But this contradicts $a \in \bigcap_{k=1}^{\infty} A_k$, which says that $a \in A_k$ for all values of $k \in \mathbb{Z}^+$. So 0 is the one and only element of $\bigcap_{k=1}^{\infty} A_k$, as required.

3. Let X and Y be any sets and $f\colon X \to Y$ any mapping. Let A and B be subsets of X. Prove that

(*i*) $f(A \cup B) = f(A) \cup f(B)$

(*ii*) $f(A \cap B) \subseteq f(A) \cap f(B)$

(*iii*) $f(B) \setminus f(A) \subseteq f(B \setminus A)$

Solution.

(*i*) Let $y \in f(A \cup B)$. Then $y = f(x)$ for some $x \in A \cup B$. Either $x \in A$, which gives $f(x) \in f(A)$, or else $x \in B$, which gives $f(x) \in f(B)$. So $y \in f(A)$ or $y \in f(B)$; that is, $y \in f(A) \cup f(B)$. As y was an arbitrary, this shows that all elements of $f(A \cup B)$ are in $f(A) \cup f(B)$; in other words, $f(A \cup B) \subseteq (f(A) \cup f(B))$.

Conversely, suppose that $y \in f(A) \cup f(B)$. Then $y \in f(A)$ or $y \in f(B)$. If $y \in f(A)$ then $y = f(x)$ for some $x \in A$; if $y \in f(B)$ then $y = f(x)$ for some $x \in B$. Since all elements of A are elements of $A \cup B$, and so are all elements of B, in either case we have $y = f(x)$ for some $x \in A \cup B$. Thus $y \in f(A \cup B)$. This holds for all $y \in f(A) \cup f(B)$; so $(f(A) \cup f(B)) \subseteq f(A \cup B)$, and since the reverse inclusion was established above, $f(A) \cup f(B) = f(A \cup B)$, as required.

(*ii*) Let $y \in f(A \cap B)$. Then $y = f(x)$ for some $x \in A \cap B$. Now $x \in A \cap B$ gives $x \in A$ and $x \in B$; so $f(x) \in f(A)$ and $f(x) \in f(B)$. Hence $y \in f(A) \cap f(B)$, and, since $y \in f(A \cap B)$ was arbitrary, this shows that $f(A \cap B) \subseteq f(A) \cap f(B)$.

We comment that the reverse inclusion does not always hold, since it is possible to have $y = f(a) = f(b)$, where a is in A and not in B, and b is in B and not in A. Under these circumstances $y \in f(A) \cap f(B)$, but $y \notin f(A \cap B)$ unless there happens to also be an element $c \in A \cap B$ such that $f(c)$ also equals y. For example, suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is the constant function $f(x) = 0$ for all x, and let $A = \{\, x \mid x > 0 \,\}$ and $B = \{\, x \mid x < 0 \,\}$. Then $f(A \cap B) = \emptyset$, but $f(A) \cap f(B) = \{0\}$.

(*iii*) Let $y \in f(B) \setminus f(A)$. Then $y \in f(B)$; so $y = f(b)$ for some $b \in B$. Furthermore, $b \notin A$, for otherwise it would follow that $y = f(b) \in f(A)$, contrary to $y \in f(B) \setminus f(A)$. So $b \in B \setminus A$, and therefore $y \in f(B \setminus A)$. Hence $f(B) \setminus f(A) \subseteq f(B \setminus A)$. (Again, the reverse inclusion fails. If $f\colon \mathbb{R} \to \mathbb{R}$ is constant then $f(\mathbb{R}) \setminus f(\{0\})$ is empty, but $f(\mathbb{R} \setminus \{0\})$ is not.)

4. Let X and Y be any sets and $f: X \to Y$ any mapping. Let A and B be subsets of Y. Prove that

(*i*) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$
(*ii*) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$
(*iii*) $f^{-1}(B) \setminus f^{-1}(A) = f^{-1}(B \setminus A)$

Solution.

First, let us emphasize that $f^{-1}(W)$, the preimage of a subset W of Y, is defined even when there is no inverse function $f^{-1}: Y \to X$. See page 3 of Choo's notes for the definitions.

(*i*) Let $x \in f^{-1}(A \cup B)$. Then $f(x) \in (A \cup B)$; so either $f(x) \in A$, giving $x \in f^{-1}(A)$, or $f(x) \in B$, giving $x \in f^{-1}(B)$. So $x \in f^{-1}(A) \cup f^{-1}(B)$.

Conversely, suppose that $x \in f^{-1}(A) \cup f^{-1}(B)$. Then either $x \in f^{-1}(A)$, giving $f(x) \in A$, or $x \in f^{-1}(B)$, giving $f(x) \in B$. So $f(x) \in A \cup B$, and therefore $x \in f^{-1}(A \cup B)$.

(*ii*) Let $x \in f^{-1}(A \cap B)$. Then $f(x) \in (A \cap B)$; so $f(x) \in A$, giving $x \in f^{-1}(A)$, and $f(x) \in B$, giving $x \in f^{-1}(B)$. So $x \in f^{-1}(A) \cap f^{-1}(B)$.

Conversely, suppose that $x \in f^{-1}(A) \cap f^{-1}(B)$. Then $x \in f^{-1}(A)$, giving $f(x) \in A$, and $x \in f^{-1}(B)$, giving $f(x) \in B$. So $f(x) \in A \cap B$, and therefore $x \in f^{-1}(A \cap B)$.

(*iii*) Let $x \in f^{-1}(B) \setminus f^{-1}(A)$. Then $x \in f^{-1}(B)$, so that $f(x) \in B$, and $x \notin f^{-1}(A)$, so that $f(x) \notin A$. So $f(x) \in B \setminus A$, which gives $x \in f^{-1}(B \setminus A)$.

Conversely, suppose that $x \in f^{-1}(B \setminus A)$. Then $f(x) \in B \setminus A$; so $f(x) \in B$, giving $x \in f^{-1}(B)$, and $f(x) \notin A$, giving $x \notin f^{-1}(A)$. So $x \in f^{-1}(B) \setminus f^{-1}(A)$.

5. Let X and Y be any sets and let $f: X \to Y$ any mapping. Let $A \subseteq X$ and $B \subseteq Y$. Prove that

(*i*) $A \subseteq f^{-1}\big(f(A)\big)$, with equality if f is injective.
(*ii*) $f\big(f^{-1}(B)\big) \subseteq B$, with equality if f is surjective.

Solution.

(*i*) Let $a \in A$. Then $f(x) \in f(A)$; and so $a \in \{\, x \mid f(x) \in f(A) \,\}$, which by definition is $f^{-1}(f(A))$. This holds for all $a \in A$; so $A \subseteq f^{-1}(f(A))$.

We now prove that the reverse inclusion holds, under the assumption that f is injective. Let $x \in f^{-1}(f(A))$. Then $f(x) \in f(A)$, and so $f(x) = f(a)$ for some $a \in A$. Since f is injective the equation $f(x) = f(a)$ yields $x = a$, and so $x \in A$. But x was chosen as an arbitrary element of $f^{-1}(f(A)$; so $f^{-1}(f(A) \subseteq A$, as required.

(*ii*) Let $y \in f(f^{-1}(B))$. Then $y = f(x)$ for some $x \in f^{-1}(B)$. But by the definition of $f^{-1}(B)$, to say that $x \in f^{-1}(B)$ is to say that $f(x) \in B$. So $y \in B$, and we have shown that $f(f^{-1}(B)) \subseteq B$.

Now assume that f is surjective, and let $b \in B$. Then $b \in Y$, the codomain of f, and so by surjectivity we have $b = f(x)$ for some $x \in X$. Since in fact $f(x) = b \in B$ we have $x \in f^{-1}(B)$. Hence $f(x) \in f(f^{-1}(B))$. That is, $b \in f(f^{-1}(B))$. So $B \subseteq f(f^{-1}(B))$.

6. (*i*) If $0 \le a \le b$, show that $\frac{a}{1+a} \le \frac{b}{1+b}$.

(*ii*) Show that if $a, b, c \ge 0$ and $a \le b + c$ then $\frac{a}{1+a} \le \frac{b}{1+b} + \frac{c}{1+c}$.

Solution.

(*i*) Given $0 \le a \le b$ we have $0 < 1+a \le 1+b$. So $1/(1+a)(1+b)$ is positive, and multiplying $1+a \le 1+b$ through by this gives $1/(1+b) \le 1/(1+a)$. So

$$\frac{a}{1+a} = 1 - \frac{1}{1+a} \le 1 - \frac{1}{1+b} = \frac{b}{1+b}.$$

(*ii*) By Part (*i*), $a/(1+a) < (b+c)/(1+b+c) = b/(1+b+c) + c/(1+b+c)$. But $0 < 1+b \le 1+b+c$; so $1/(1+b+c) \le 1/(1+b)$, and so $b/(1+b+c) \le b/(1+b)$, Similarly, $0 < 1+c \le 1+b+c$ gives $c/(1+b+c) \le c/(1+c)$. So

$$\frac{a}{1+a} \le \frac{b}{1+b+c} + \frac{c}{1+b+c} \le \frac{b}{1+b} + \frac{c}{1+c},$$

as required.

7. Let (X, d) and (Y, d') be metric spaces, $f: X \to Y$ a function and $x_0 \in X$. Show that the following two conditions are equivalent:

For each open ball B with centre at $f(x_0)$ there is an open ball C with centre at x_0 such that $C \subseteq f^{-1}(B)$;

For each open set U with $x_0 \in f^{-1}(U)$ there is an open ball C with centre at x_0 such that $C \subseteq f^{-1}(U)$.

Solution.

Suppose that the first of these two conditions hold; we prove that the second holds too. Let U be an arbitrary open set with $x_0 \in f^{-1}(U)$. Then $f(x_0) \in U$, and since U is open there is an open ball B with centre at $f(x_0)$ such that $B \subseteq U$. By our given condition there is an open ball C with centre at x_0 such that $C \subseteq f^{-1}(B)$. Now for each $x \in C$ we have $x \in f^{-1}(B)$, and hence $f(x) \in B \subseteq U$. So $x \in f^{-1}(U)$, and since this holds for all $x \in C$ it follows that $C \subseteq f^{-1}(U)$. Thus we have shown, as required, that for every open set U with $x_0 \in f^{-1}(U)$ there is an open ball C centred at x_0 such that $C \subseteq f^{-1}(B)$.

Conversely, suppose that the second of the two condition holds, and let B be an open ball with centre at $f(x_0)$. Then $f(x_0) \in B$, and so $x_0 \in f^{-1}(B)$. Since open balls are open sets, we see that $U = B$ is an open set with $x_0 \in f^{-1}(U)$, and by the given condition there is an open ball C centred at x_0 with $C \subseteq f^{-1}(U) = f^{-1}(B)$. So we have shown, as required, that for each open ball B with centre at $f(x_0)$ there is an open ball C with centre at x_0 such that $C \subseteq f^{-1}(B)$.