Week 3 Summary

Lecture 5

The rule for finding the continued fraction expansion of a number is this. Find the integer part of the number, then subtract off this integer part and invert what is left. Repeat these steps. The sequence of integer parts that you get is the sequence of partial quotients of the continued fraction for the number you started with.

This works nicely for the continued fraction of $\sqrt{d}$, whenever d is a positive integer that is not a square. It turns out that in this case the continued fraction has a repeating block of partial quotients. In fact

$$\sqrt{d} = [a_1, a_2, a_3, \ldots, a_k, 2a_1, a_2, a_3, \ldots, a_k, 2a_1, a_2, a_3, \ldots]$$

for some positive integers a_i. Note that you do not need to use a calculator and get an approximation to the decimal expansion of $\sqrt{d}$ in order to find the a_i. You can do it exactly. For example, since $4 < \sqrt{19} < 5$, the integer part of $\sqrt{19}$ is 4. Now

$$\frac{1}{\sqrt{19}-4} = \frac{\sqrt{19}+4}{19-4^2} = 2 + \frac{\sqrt{19}-2}{3},$$

where again we used $4 < \sqrt{19} < 5$ to see that 2 is the integer part of $\frac{\sqrt{19}+4}{3}$. Subtracting 2 and inverting gives

$$\frac{3}{\sqrt{19}-2} = \frac{3(\sqrt{19}+2)}{19-2^2} = \frac{\sqrt{19}+2}{5} = 1 + \frac{\sqrt{19}-3}{5},$$

and so on. We find $\sqrt{19} = [4, \overline{2, 1, 3, 1, 2, 8}]$, where the overlined part repeats.

Let $a_1, a_2, a_3, \ldots$ be any infinite sequence of positive integers, and consider the corresponding Magic Table:

$$\begin{array}{cc|ccccc} & & a_1 & a_2 & a_3 & a_4 & \cdots \\ \hline 0 & 1 & p_1 & p_2 & p_3 & p_4 & \cdots \\ 1 & 0 & q_1 & q_2 & q_3 & q_4 & \cdots \end{array}.$$

It is clear that the q_i's form a strictly increasing sequence of positive integers. If $a_i = 1$ for all i we find that $q_i = F_i$, the i-th Fibonacci number, and $p_i = F_{i+1}$. It is easy to prove by induction on i that for any sequence $a_1, a_2, a_3, \ldots$, the numbers q_i in the Magic Table satisfy $q_i \geq F_i$ for all i. The numbers get big quite quickly.

We have shown that $p_i q_{i+2} - q_i p_{i+2} = (-1)^i a_{i+2}$. Dividing through by $q_i q_{i+2}$ we see that

$$\frac{p_i}{q_i} - \frac{p_{i+1}}{q_{i+1}} = \frac{(-1)^i a_{i+2}}{q_i q_{i+2}}$$

which is positive if i is even, negative if i is odd. Thus we have the following proposition.

***Proposition:** The odd-numbered convergents $(\frac{p_1}{q_1}, \frac{p_3}{q_3}, \frac{p_5}{q_3}, \ldots)$ form an increasing sequence, while the even-numbered ones $(\frac{p_2}{q_2}, \frac{p_4}{q_4}, \frac{p_6}{q_6}, \ldots)$ form a decreasing sequence.

Lecture 6

Continue the notation from Lecture 5. Recall that we have shown that

$$p_i q_{i+1} - q_i p_{i+1} = (-1)^i.$$

Note as a consequence of this that p_i and q_i are coprime: if there were any $d > 1$ that were a factor of both p_i and q_i then it would also be a factor of $p_i q_{i+1} - q_i p_{i+1} = (-1)^i$—which is clearly impossible. So the rational number $\frac{p_i}{q_i}$ is in its lowest terms.

Note that $\frac{p_i}{q_i} - \frac{p_{i+1}}{q_{i+1}} = \frac{(-1)^i}{q_i q_{i+1}}$, which is positive if $i > 0$, negative if $i < 0$. So each odd numbered convergent is less than the adjacent even-numbered convergents.

***Proposition:** If i is odd and j is even then $\frac{p_i}{q_i} < \frac{p_j}{q_j}$.

To prove this one considers separately the cases $i < j$ and $i > j$. In the former case we have $\frac{p_i}{q_i} < \frac{p_{i+2}}{q_{i+2}} < \cdots < \frac{p_{j-1}}{q_{j-1}}$ (since the odd-numbered convergents form an increasing sequence) and $\frac{p_{j-1}}{q_{j-1}} < \frac{p_j}{q_j}$ (odd-numbered convergent less than adjacent even-numbered convergent). In the case $i > j$ we have $\frac{p_j}{q_j} > \frac{p_{j+2}}{q_{j+2}} < \cdots < \frac{p_{i-1}}{q_{i-1}}$ (since the even-numbered convergents form a decreasing sequence) and $\frac{p_{i-1}}{q_{i-1}} > \frac{p_i}{q_i}$ (even-numbered convergent greater than adjacent odd-numbered convergent).

It follows from this that the sequence of odd-numbered convergents, as well as being increasing, is bounded above (by every even-numbered convergent. So the odd-numbered convergents approach some limit α^-. Similarly, the sequence of even-numbered convergents is decreasing and bounded below (by each odd-numbered convergent). So this sequence also approaches some limit, α^+ (say). Now

$$\alpha^+ - \alpha^- = \lim_{k\to\infty} \frac{p_{2k}}{q_{2k}} - \lim_{k\to\infty} \frac{p_{2k+1}}{q_{2k+1}} = \lim_{k\to\infty} \left(\frac{p_{2k}}{q_{2k}} - \frac{p_{2k+1}}{q_{2k+1}}\right) = \lim_{k\to\infty} \frac{1}{q_{2k} q_{2k+1}} = 0.$$

So $\alpha^+ = \alpha^-$. The even and odd numbered convergents approach the same limit α. For every k, the limit α lies between $\frac{p_k}{q_k}$ and $\frac{p_{k+1}}{q_{k+1}}$. So

$$\left|\frac{p_k}{q_k} - \alpha\right| < \left|\frac{p_k}{q_k} - \frac{p_{k+1}}{q_{k+1}}\right| = \frac{1}{q_k q_{k+1}} < \frac{1}{q_k^2}.$$

We conclude that if $\frac{p}{q}$, in its lowest terms, is one of the convergents of the continued fraction expansion of α, then $|\frac{p}{q} - \alpha| < \frac{1}{q^2}$. If α is irrational, this gives infinitely many rational numbers $\frac{p}{q}$ such that $|\frac{p}{q} - \alpha| < \frac{1}{q^2}$. But if is α is rational, its continued fraction terminates, and, indeed, we can prove the following proposition.

***Proposition** If α is rational there are only finitely many rational numbers $\frac{p}{q}$ such that $|\frac{p}{q} - \alpha| < \frac{1}{q^2}$.

The general proof of this follows the same lines as the examples in Exercise 3 of Tutorial 2. Writing $\alpha = \frac{a}{b}$, where $a, b \in \mathbb{Z}$ and $b > 0$, it can be seen that

$|\frac{p}{q} - \alpha| < \frac{1}{q^2}$ is equivalent to $|pb - qa| < \frac{b}{q}$ (assuming $q > 0$—which involves no loss of generality). In the case $q \geq b$ the only solution of this is $pb - qa = 0$, which gives $\frac{p}{q} = \frac{a}{b} = \alpha$. For each $q < b$ there are only finitely many solutions: they are given by values of p such that $qa - \frac{b}{q} < pb < qa + \frac{b}{q}$, and there are only a finite number of multiples of b between $qa - \frac{b}{q}$ and $qa + \frac{b}{q}$.

We now leave continued fractions for a while, and study certain Diophantine equations. (That is, equations where the unknowns are integers.)
First, let us consider the equation $xa + yb = c$, where a, b and c are given integers, and x and y are the unknowns.
If the gcd of a and b is not a divisor of c there are no solutions (since every common divisor of a and b is a divisor of $xa + yb$ for all integers x and y). So let $d = \gcd(a, b)$, and suppose that $c = dk$ for some integer k. The Euclidean Agorithm can be used to find integers p and q such that $pa + qb = d$, and now multiplying through by k we see that $x_0 = pk$ and $y_0 = qk$ gives one solution of $xa + yb = dk$. If M is any common multiple of a and b then $x = x_0 + \frac{M}{a}$ and $y = y_0 - \frac{M}{b}$ is also a solution, since these formulas give

$$xa + yb = (x_0 + \frac{M}{a})a + (y_0 - \frac{M}{b})b = x_0a + M + y_0b - M = x_0a + y_0b = c.$$

It is also true that every solution x, y has this form. For suppose that $xa + yb = c$. Then $xa + yb = x_0a + y_0b$, and so $xa - x_0a = y_0b - yb = M$ (say). This number M is a multiple of a (since $M = (x - x_0)a$) and of b (since $M = (y_0 - y)b$); furthermore, rearranging the equtions gives $x = x_0 + \frac{M}{a}$ and $y = y_0 - \frac{M}{b}$.

(The material dealt with this week corresponds quite closely to pages 88–94 and 30–32 of Walters' book.)