The University of Sydney

MATH2902 Vector Spaces

(http://www.maths.usyd.edu.au/u/UG/IM/MATH2902/)

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Tutorial 5

1. In each case decide whether or not the set S is a vector space over the field F, relative to obvious operations of addition and scalar multiplication. If it is, decide whether it has finite dimension, and if so, find the dimension.

(*i*) $S = \mathbb{C}$ (complex numbers), $F = \mathbb{R}$.

(*ii*) $S = \mathbb{C}$, $F = \mathbb{C}$.

(*iii*) $S = \mathbb{R}$, $F = \mathbb{Q}$ (rational numbers).

(*iv*) $S = \mathbb{R}[X]$ (polynomials over $\mathbb{R}$ in the variable X—that is, expressions of the form $a_0 + a_1X + \cdots + a_nX^n$ $(a_i \in \mathbb{R})$), $F = \mathbb{R}$.

(*v*) $S = \mathrm{Mat}(n, \mathbb{C})$ ($n \times n$ matrices over $\mathbb{C}$), $F = \mathbb{R}$.

2. Let $\mathbb{Z}_2$ be the field which has just the two elements 0 and 1. (See §1d#10 of the book.) How many elements will there be in a four dimensional vector space over $\mathbb{Z}_2$?

3. (*i*) Let V be a vector space over a field F and let S be any set. Convince yourself that that the set of all functions from S to V becomes a vector space over F if addition and scalar multiplication of functions are defined in the usual way.

(Hint: To do this in detail requires checking that all the vector space axioms are satisfied. However, the proof in §3b#6 of the book is almost word for word the same as the proof required here.)

(*ii*) Use part (*i*) to show that if V and W are both vector spaces then the set of all linear transformations from V to W is a vector space (with the usual definitions of addition and scalar multiplication of functions).

4. Let U and V be vector spaces over a field F. A function $f\colon V \to W$ is called a *vector space isomorphism* if f is a bijective linear transformation. Prove that if $f\colon U \to V$ is a vector space isomorphism then the inverse function $f^{-1}\colon V \to U$ (defined by the rule that $f^{-1}(v) = u$ if and only if $f(u) = v$) is also a vector space isomorphism.

5. (*i*) Prove that if v_1, v_2, ... , v_n are linearly independent elements of a vector space V and $v_{n+1} \in V$ is not contained in $\text{Span}(v_1, v_2, \ldots, v_n)$ then v_1, v_2, ... , v_{n+1} are linearly independent.

(*ii*) If v_1, v_2, ... , v_n are linearly independent elements of V and V is spanned by elements w_1, w_2, ... , w_m then $n \leq m$. (This is Theorem 4.14 of the book, the proof of which was relatively hard.) Use this result and the first part to prove that if v_1, v_2, ... , v_n are linearly independent then there exist v_{n+1}, v_{n+2}, ... , $v_d \in V$ such that v_1, v_2, ... , v_d form a basis of V.