THE UNIVERSITY OF SYDNEY

MATH2902 Vector Spaces

(http://www.maths.usyd.edu.au/u/UG/IM/MATH2902/)

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Tutorial 2

1. Let A be a 4×4 matrix, and suppose that v_1, v_2, v_3 and v_4 are column vectors satisfying $Av_1 = 2v_1$, $Av_2 = 2v_2 + v_1$, $Av_3 = 3v_3$ and $Av_4 = 3v_4 + v_3$. Let T be the matrix whose columns are v_1, v_2, v_3 and v_4 (in that order). Prove that

$$AT = T\begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

Solution.

Define

$$J = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

Observe that J is the result of applying the following sequence of elementary column operations to a 4×4 identity matrix: multiply the second column by 2, add the first column to the second, multiply the first column by 2, multiply the fourth column by 3, add the third column to the fourth, multiply the third column by 3. So TJ must be the result of applying the same sequence of elementary column operations to T. Hence the columns of TJ are $2v_1$, $2v_2 + v_1$, $3v_3$ and $3v_4 + v_3$.

This same result can also be seen by multiplication of partitioned matrices. The first column of TJ is obtained by multiplying T by the first column of J. We have $T = \begin{pmatrix} v_1 & | & v_2 & | & v_3 & | & v_4 \end{pmatrix}$, and so the first column of TJ is

$$\begin{pmatrix} v_1 & | & v_2 & | & v_3 & | & v_4 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 2v_1 + 0v_2 + 0v_3 + 0v_4 = 2v_1.$$

Similarly the second column of TJ is $v_1 + 2v_2 + 0v_3 + 0v_4$, and the third and fourth columns can also be checked easily.

As for the other side of the equation, we know that the first column of AT is Av_1 (since v_1 is the first column of T), and we are given that this is $2v_1$. Similarly, the second column of AT is Av_2, which equals $2v_2 + v_1$, the third column of AT is $Av_3 = 3v_3$ and the last column of AT is $Av_4 = 3v_4 + v_3$. So $AT = TJ$, as required.

2. For each of the following matrices A find a nonsingular matrix T such that $T^{-1}AT$ is diagonal.

(a) $A = \begin{pmatrix} 9 & -2 & 7 \\ 4 & -1 & 4 \\ -4 & 2 & -2 \end{pmatrix}$ (b) $A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

Check that it is possible in part (b) to choose T in such a way that the sum of the squares of the entries in each column of T is 1, and that if this is done then $T^{-1} = {}^{\mathrm{t}}T$.

Solution.

(a) The first step is to find the values of x for which $\det(A - xI) = 0$. We have

$$\begin{aligned} \det(A - xI) &= (9-x)((-1-x)(-2-x) - 8) + 2(4(-2-x) + 16) \\ &\qquad + 7(8 + 4(-1-x)) \\ &= (9-x)(x^2 + 3x - 6) + 2(-4x + 8) + 7(-4x + 4) \\ &= -x^3 + 6x^2 + 33x - 54 - 8x + 16 - 28x + 28 \\ &= -(x^3 - 6x^2 + 3x + 10) \\ &= -(x+1)(x-2)(x-5), \end{aligned}$$

so that the eigenvalues are -1, 2 and 5.

Next we must find an eigenvector for each of the eigenvalues; to do this we must solve $(A+I)u = 0$, $(A-2I)v = 0$ and $(A-5I)w = 0$. Applying row operations to $A + I$ we obtain the reduced echelon matrix

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{pmatrix}$$

enabling a (-1)-eigenvector to be readily calculated. The calculations for 2 and 5 are similar, and we find that

$$u = \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}, \qquad v = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \qquad w = \begin{pmatrix} 17 \\ 6 \\ -8 \end{pmatrix}$$

are eigenvectors. (Note that any nonzero scalar multiples of these would do equally well.) The matrix T which has u, v and w as its columns is a suitable diagonalizing matrix.

(b) Using the same procedure as above, we have

$$\begin{aligned}\det(A - xI) &= (2-x)(-x(2-x)-1) + 1(x-1) + 1(-1-(2-x))\\ &= (2-x)(x^2-2x-1) + x - 1 + x - 3\\ &= -(x^3 - 4x^2 + x + 6)\\ &= -(x+1)(x-2)(x-3).\end{aligned}$$

Three eigenvectors (corresponding to the eigenvalues -1, 2 and 3 respectively) are

$$\begin{pmatrix}1\\1\\-2\end{pmatrix}, \quad \begin{pmatrix}1\\1\\1\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}1\\-1\\0\end{pmatrix},$$

so that

$$T = \begin{pmatrix}1 & 1 & 1\\ 1 & 1 & -1\\ -2 & 1 & 0\end{pmatrix}$$

is a suitable diagonalizing matrix.

The sums of the squares of the elements in the columns can be made equal to 1 by dividing the entries in the first column by $\sqrt{6}$, the entries in the second column by $\sqrt{3}$ and the entries in the last column by $\sqrt{2}$. It is clear that if this is done then the diagonal entries of $({}^{t}T)T$ are all equal to 1. An easy computation verifies that the off-diagonal entries are all 0.

3. Prove that if A and B are matrices such that AB is defined then ${}^{t}B\,{}^{t}A$ is defined, and ${}^{t}B\,{}^{t}A = {}^{t}(AB)$.

Solution.

Since AB is defined the number of columns of A equals the number of rows of B. So the number of columns of ${}^{t}B$ (which equals the number of rows of B) equals the number of rows of ${}^{t}A$ (which equals the number of columns of A). Hence ${}^{t}B\,{}^{t}A$ is defined.

Let A have shape $m \times n$ and B shape $n \times p$. Then ${}^{t}(AB)$ and ${}^{t}B\,{}^{t}A$ are both $p \times m$ matrices. Let i and j be arbitrary subject to $1 \le i \le p$ and $1 \le j \le m$. Then the (i, j)-entry of ${}^{t}B\,{}^{t}A$ is

$$({}^{t}B\,{}^{t}A)_{ij} = \sum_{k=1}^{n} ({}^{t}B)_{ik}({}^{t}A)_{kj} = \sum_{k=1}^{n} B_{ki}A_{jk} = \sum_{k=1}^{n} A_{jk}B_{ki} = (AB)_{ji} = ({}^{t}(AB))_{ij}.$$

Hence ${}^{t}(AB) = {}^{t}B\,{}^{t}A$, as required.

4. Let A be a matrix satisfying ${}^{t}A = A$ and let u and v be eigenvectors of A with corresponding eigenvalues λ and μ. (That is, u and v are nonzero and $Au = \lambda u$ and $Av = \mu v$.) Prove that if $\lambda \ne \mu$ then $({}^{t}u)v = 0$. (Hint: Show that $({}^{t}u)A = \lambda({}^{t}u)$, and then expand $({}^{t}u)Av$ in two ways.)

Investigate the connection between this exercise and 2 (b).

Solution.

Since transposing reverses products and since $Au = \lambda u$, we have

$$({}^{t}u)A = ({}^{t}u)({}^{t}A) = {}^{t}(Au) = {}^{t}(\lambda u) = \lambda({}^{t}u).$$

Hence

$$\lambda({}^{t}u)v = \big(({}^{t}u)A\big)v = ({}^{t}u)(Av) = ({}^{t}u)(\mu v) = \mu({}^{t}u)v,$$

and since $\lambda \ne \mu$ it follows that $({}^{t}u)v = 0$.

If the columns of T are u, v and w then the off-diagonal entries of $({}^{t}T)T$ are $({}^{t}u)v$ and the five other similar expressions. This exercise shows that if u, v and w are eigenvectors of a symmetric matrix corresponding to eigenvalues which are distinct then the off-diagonal entries of $({}^{t}T)T$ are zero.

5. Show that if α and β are arbitrary complex numbers then $\overline{(\alpha+\beta)} = \overline{\alpha} + \overline{\beta}$ and $\overline{\alpha\beta} = \overline{\alpha}\,\overline{\beta}$, where the overline denotes complex conjugation (defined by $\overline{(x+iy)} = x - iy$ for all $x, y \in \mathbb{R}$, where $i = \sqrt{-1}$).

If A is a complex matrix let $\overline{A}$ be the matrix whose entries are the complex conjugates of the entries of A. Use the previous part to show that $\overline{AB} = \overline{A}\,\overline{B}$ for all complex matrices A and B such that AB exists.

Solution.

Let $\alpha = x + iy$ and $\beta = u + iv$, where $x, y, u, v \in \mathbb{R}$. Then

$$\begin{aligned}\overline{\alpha\beta} = \overline{(x+iy)(u+iv)} &= \overline{(xu - yv) + i(xv + yu)}\\ &= (xu - yv) - i(xv + yu) = (x - iy)(u - iv) = \overline{\alpha}\overline{\beta}.\end{aligned}$$

Similarly $\overline{\alpha+\beta} = \overline{(x+u) + i(y+v)} = (x+u) - i(y+v) = \overline{\alpha} + \overline{\beta}$.

Let $A \in \mathrm{Mat}(m \times n, \mathbb{C})$ and $B \in \mathrm{Mat}(n \times p, \mathbb{C})$. If $1 \le r \le m$ and $1 \le s \le p$ then

$$\begin{aligned}(\overline{AB})_{rs} = \overline{(AB)_{rs}} &= \overline{\sum_{t=1}^{n} A_{rt}B_{ts}} = \sum_{t=1}^{n} \overline{A_{rt}B_{ts}}\\ &= \sum_{t=1}^{n} \overline{A_{rt}}\,\overline{B_{ts}} = \sum_{t=1}^{n} (\overline{A})_{rt}(\overline{B})_{ts}) = (\overline{A}\,\overline{B})_{rs},\end{aligned}$$

and it follows that $\overline{AB} = \overline{A}\,\overline{B}$.