## Normal subgroups

Let $G$ be a group and $K$ a subgroup of $G$. If $t \in G$ then we define

$$tKt^{-1} = \{\, tkt^{-1} \mid k \in K \,\}.$$

It is easy to show that $tKt^{-1}$ is also a subgroup of $G$. The proof is included here for completeness, although it was not given in lectures and is therefore not examinable.

Since $K$ is a subgroup of $G$ the identity element $e$ of $G$ is contained in $K$ (since $K$ satisfies SG2). So $tet^{-1} \in tKt^{-1}$. But $tet^{-1} = tt^{-1} = e$, and so $e \in tKt^{-1}$. Hence $tKt^{-1}$ satisfies SG2.

Let $x$, $y$ be arbitrary elements of $tKt^{-1}$. Then $x = tht^{-1}$ and $y = tkt^{-1}$ for some $h$, $k \in K$, and this gives

$$xy = (tht^{-1})(tkt^{-1}) = th(t^{-1}t)kt^{-1} = thekt^{-1} = t(hk)t^{-1}.$$

Since $K$ satisfies SG1 and $h$, $k \in K$ it follows that $hk \in K$, and so $t(hk)t^{-1} \in tKt^{-1}$. Hence we have shown that $xy \in tKt^{-1}$ whenever $x$, $y \in tKt^{-1}$. Thus $tKt^{-1}$ satisfies SG1.

Let $x \in tKt^{-1}$ be arbitrary. Then $x = tkt^{-1}$ for some $k \in K$, and since inverting reverses products it follows that $x^{-1} = (t^{-1})^{-1}k^{-1}t^{-1} = tk^{-1}t^{-1}$. Since $k \in K$ and $K$ satisfies SG3 it follows that $k^{-1} \in K$, and hence $tk^{-1}t^{-1} \in tKt^{-1}$. So we have shown that $x^{-1} \in tKt^{-1}$ whenever $x \in tKt^{-1}$. Thus $tKt^{-1}$ satisfies SG3, as required.

If $G$ is Abelian then $tkt^{-1} = ktt^{-1} = ke = k$, and so in this case $tKt^{-1} = K$. But if $G$ is not Abelian then it is quite possible that $tKt^{-1} \neq K$. It is also quite possible to have $tKt^{-1} = K$ even when $G$ is not Abelian.

**Definition.** A subgroup $K$ of a group $G$ is said to be a *normal subgroup* if $tKt^{-1} = K$ for all $t \in G$.

For example, suppose that $\phi\colon G \to H$ is a group homomorphism, and let $K$ be the kernel of $\phi$. That is,

$$K = \{\, g \in G \mid \phi(g) = e_H \,\},$$

where $e_H$ is the identity element of $H$. We proved last week that $K$ is a subgroup of $G$; in fact, it is easy to see that this subgroup is necessarily normal.

Indeed, let $t$ be an arbitrary element of $G$, and suppose that $x \in K$. Then $\phi(x) = e_H$, and so

$$\begin{aligned}
\phi(txt^{-1}) &= \phi(t)\phi(x)\phi(t^{-1}) && \text{(since } \phi \text{ is a homomorphism)}\\
&= \phi(t)e_H\phi(t^{-1}) && \text{(since } x \in K\text{)}\\
&= \phi(t)\phi(t^{-1}) && \text{(since } he_H = h \text{ for all } h \in H\text{)}\\
&= \phi(tt^{-1}) && \text{(since } \phi \text{ is a homomorphism)}\\
&= \phi(e_G) && \text{(by the definition of } t^{-1}\text{)}\\
&= e_H,
\end{aligned}$$

since we showed last week that $\phi(e_G) = e_H$. The above calculation shows that $txt^{-1} \in K$ whenever $x \in K$, and a similar calculation shows that $x \in K$ whenever $txt^{-1} \in K$. Hence the elements of $tKt^{-1}$ are the same as the elements of $K$. So we have shown that $tKt^{-1} = K$ for all $t \in G$, as required.

The result we have just proved deserves to be called a theorem:

**Theorem.** *Kernels of homomorphisms are normal subgroups.*

It is in fact true that every normal subgroup of a group $G$ is the kernel of some homomorphism defined on $G$; so we can say that normal subgroups and kernels are the same things.

Note that $tKt^{-1} = K$ is equivalent to $tK = Kt$. Thus to say that a subgroup $K$ is normal is to say that the right cosets of $K$ and the left cosets of $K$ are the same as each other.

It must be emphasized that the equation $tK = Kt$ does not mean that $tk = kt$ for all $k \in K$; rather, what it means is that for all $k \in K$ there exists an $h \in K$ such that $tk = ht$.

Assume that $K$ is a normal subgroup of $G$, and define a relation $\equiv_K$ on $G$ as follows:

$$x \equiv_H y \text{ if and only if } x = ky \text{ for some } k \in K.$$

We have shown that this is an equivalence relation. Indeed, in our discussion of cosets in Week 8 we observed that the right cosets of $K$ are precisely the equivalence classes for the relation $\equiv_K$. Because $K$ is normal we can equally well say that the equivalence classes are left cosets of $K$.

The equivalence relation $\equiv_K$ is important even when the subgroup $K$ is not assumed to be normal, but when $K$ is normal it is especially important. The crucial fact which makes this so is as follows.

**Proposition.** *Let $K$ be a normal subgroup of $G$ and let $\equiv_K$ be as defined above. Let $x$, $y$, $x'$, $y' \in G$. If $x' \equiv_K x$ and $y' \equiv_K y$ then $x'y' \equiv_K xy$.*

*Proof.* If $x' \equiv_K x$ and $y' \equiv_K y$ then $x' = hx$ and $y' = ky$ for some $h$, $k \in K$. Now this gives $x'y' = (hx)(ky) = h(xk)y$, and since $xK = Kx$ we know that $xk = lx$ for some $l \in K$. Thus $x'y' = hlxy$, with $h$, $l \in K$, and since $K$ is closed under multiplication this shows that $x'y' \equiv_K xy$, as required. □

The proposition tells us that if $x'$ is in the same $\equiv_K$-equivalence class as $x$ and $y'$ is in the same $\equiv_K$-equivalence class as $y$ then $x'y'$ is in the same $\equiv_K$-equivalence class as $xy$. Thus we can define multiplication of $\equiv_K$-equivalence classes in a sensible way: if $A$, $B$ are $\equiv_K$-equivalence classes then $AB$ is defined to be the $\equiv_K$-equivalence class that contains all the products $xy$ for $x \in A$ and $y \in B$.

Since the $\equiv_K$-equivalence classes are simply the cosets of $K$ in $G$, the equivalence class that contains $x$ is $Kx$. So our rule for multiplication of equivalence classes can be stated as follows: $(Kx)(Ky) = Kxy$, for all $x$, $y \in G$.

For example, let $G = \mathrm{Sym}(4)$ and let $K = \{\,\mathrm{id},\ (1,2)(3,4),\ (1,3)(2,4),\ (1,4)(2,3)\,\}$. We know that $K$ is a normal subgroup of $G$ since, as we saw last week, it is the kernel of the homomorphism $\phi\colon \mathrm{Sym}(4) \to \mathrm{Sym}(3)$ that we have investigated so carefully. The equivalence relation $\equiv_K$ is exactly the same as the equivalence relation we investigated last week: if $x$, $y \in \mathrm{Sym}(4)$ then $x \equiv_K y$ if and only if $\phi(x) = \phi(y)$. So the cosets of $K$ in $G$ are the equivalence classes $S_1$, $S_2$, $S_3$, $S_4$, $S_5$ and $S_6$ defined in the notes for Week 12, namely

$$\begin{aligned}
S_1 &= \{\mathrm{id},\ (1,2)(3,4),\ (1,3)(2,4),\ (1,4)(2,3)\},\\
S_2 &= \{(1,2),\ (3,4),\ (1,4,2,3),\ (1,3,2,4)\},\\
S_3 &= \{(2,3),\ (1,2,4,3),\ (1,3,4,2),\ (1,4)\},\\
S_4 &= \{(1,3),\ (1,4,3,2),\ (2,4),\ (1,2,3,4)\},\\
S_5 &= \{(1,2,3),\ (2,4,3),\ (1,4,2),\ (1,3,4)\},\\
S_6 &= \{(1,3,2),\ (1,4,3),\ (2,3,4),\ (1,2,4)\}.
\end{aligned}$$

Here $S_1 = K$, and, for example, $S_2 = K(1,2)$. You should check that multiplying the four elements of $K$ by $(1,2)$ gives the four elements of $S_2$. Note that it is equally true that $S_2 = K(3,4) = K(1,4,2,3) = K(1,3,2,4)$: we can choose any element of the coset as a representative of the coset and our formulas will remain valid.

The rule that $(Kx)(Ky) = Kxy$ tells us, for example, that $S_2S_3 = S_6$, since if $x \in S_2$ and $y \in S_3$ then $xy \in S_6$. It does not matter which elements of $S_2$ and $S_3$ are chosen for $x$ and $y$: for example, $(1,2)(2,3) = (1,3,2) \in S_6$, and $(3,4)(1,4) = (1,4,3) \in S_6$.

We conclude the course by stating the two important theorems of group theory whose proofs are essentially incorporated in the discussion above and in the Week 12 notes.

**Theorem.** *Let $G$ be a group and $K$ a normal subgroup of $G$. Let $G/K \stackrel{\text{def}}{=} \{\, Kx \mid x \in G \,\}$, the set of all cosets of $K$ in $G$. Then there is a multiplication operation on $G/K$ such that $(Kx)(Ky) = Kxy$ for all $x$, $y \in G$, and $G/K$ is a group with respect to this multiplication.*

The group $G/K$ defined in the statement of this theorem is called the *quotient group* of $G$ modulo the normal subgroup $K$.

**The Fundamental Homomorphism Theorem.** *Suppose that $\phi: G \to H$ is a group homomorphism. Let $K$ be the kernel of $\phi$ and $I$ the image of $\phi$. Then $I$ is a subgroup of $H$ and $K$ is a normal subgroup of $G$; furthermore, the quotient group $G/K$ is isomorphic to $I$.*