THE UNIVERSITY OF SYDNEY

MATH2008 Introduction to Modern Algebra

(http://www.maths.usyd.edu.au/u/UG/IM/MATH2008/)

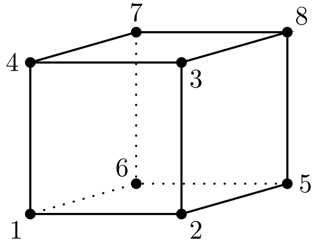
Semester 2, 2003 — Lecturer: R. Howlett

Tutorial 8

1. Draw a cube and label its vertices with the numbers 1, 2, ..., 8. Choose the labelling so that 1, 2, 3, and 4 are the corners of one face, with 1 adjacent to 2 and 4, and so that 8 is opposite to 1, 7 opposite to 2, 6 opposite to 3 and 5 opposite to 4. (The line joining opposite vertices must pass through the central point of the cube.)

Write down as many rotations of the cube as you can find, representing them as elements of Sym(8). (For example, $(1,4)(5,8)(2,7)(3,6)$ is a rotation through 180° about an axis that joins the midpoints of the edges 14 and 85.)

Solution.



The identity element, id, counts as a trivial rotation (through zero degrees, about any axis). There are nontrivial rotational symmetries corresponding to the following axes:

(a) from the central point of the face 1234 to the central point of the face 5678; rotations through 90°, 180° and 270°, corresponding to the permutations $(1,2,3,4)(8,7,6,5)$, $(1,3)(2,4)(8,6)(7,5)$ and $(4,3,2,1)(5,6,7,8)$;

(b) from the central point of the face 1674 to the central point of the face 2583; rotations through 90°, 180° and 270°, corresponding to the permutations $(1,6,7,4)(2,5,8,3)$, $(1,7)(6,4)(8,2)(3,5)$ and $(1,4,7,6)(2,3,8,5)$;

(c) from the central point of the face 1256 to the central point of the face 4387; rotations through 90°, 180° and 270°, corresponding to the permutations $(1,2,5,6)(4,3,8,7)$, $(1,5)(2,6)(4,8)(3,7)$ and $(1,6,5,2)(4,7,8,3)$;

(d) from vertex 1 to vertex 8; rotations through 120° and 240°, corresponding to the permutations $(2,4,6)(7,5,3)$, and $(2,6,4)(7,3,5)$;

(e) from vertex 2 to vertex 7; rotations through 120° and 240°, corresponding to the permutations $(1,3,5)(8,6,4)$, and $(1,5,3)(8,4,6)$;

(f) from vertex 3 to vertex 6; rotations through 120° and 240°, corresponding to the permutations $(2,4,8)(7,5,1)$, and $(2,8,4)(7,1,5)$;

(g) from vertex 4 to vertex 5; rotations through 120° and 240°, corresponding to the permutations $(1,3,7)(8,6,2)$, and $(1,7,3)(8,2,6)$;

(h) from the midpoint of the edge 14 to the midpoint of the edge 85; a half-turn, corresponding to the permutation $(1,4)(5,8)(2,7)(3,6)$;

(i) from the midpoint of the edge 43 to the midpoint of the edge 56; a half-turn, corresponding to the permutation $(3,4)(5,6)(2,7)(1,8)$;

(j) from the midpoint of the edge 32 to the midpoint of the edge 67; a half-turn, corresponding to the permutation $(3,2)(7,6)(4,5)(1,8)$;

(k) from the midpoint of the edge 21 to the midpoint of the edge 78; a half-turn, corresponding to the permutation $(1,2)(7,8)(4,5)(3,6)$;

(l) from the midpoint of the edge 16 to the midpoint of the edge 38; a half-turn, corresponding to the permutation $(1,6)(3,8)(4,5)(2,7)$;

(m) from the midpoint of the edge 25 to the midpoint of the edge 74; a half-turn, corresponding to the permutation $(2,5)(7,4)(3,6)(1,8)$.

Altogether this makes 24 rotations. They form a group, which is known as the *octahedral group*, since it is also the group of rotational symmetries of a regular octahedron.

2. Determine which of the following permutations are even, and which are odd: $(1,2,3)$; $(1,2)(3,4)(5,6)$; $(1,6,2,4,9)(7,8,3)$; $(1,2,3,\ldots,2000)$. Determine also the orders of these permutations.

Solution.

Since odd length cycles are even and even length cycles are odd, a permutation is odd if and only if it has an odd number of cycles of even length. So $(1,2,3)$ is even (no even length cycles), $(1,2)(3,4)(5,6)$ is odd (3 even length cycles), $(1,6,2,4,9)(7,8,3)$ is even (no even length cycles) and $(1,2,3,\ldots,2000)$ is odd (1 even length cycle).

The order of a permutation is the least common multiple of the lengths of its cycles. So the orders of the above four are, respectively, 3, 2, 15 and 2000.

3. Write down all the even permutations in Sym(4).

Solution.

There must be an even number of cycles of even length to make the permutation even. In Sym(4) the longest possible cycle length is 4, and it is not possible to have more than one of these. It is possible to have two cycles of length 2: this gives the three permutations $(1,2)(3,4)$, $(1,3)(2,4)$ and $(1,4)(2,3)$. The only other possibility is to have no even length cycles. This can be done by having one 3-cycle and one 1-cycle (e.g. $(1,2,3)(4)$), or by having four 1-cycles. Four 1-cycles means four fixed points, and the only permutation of $\{1,2,3,4\}$ that fixes everything is the identity. There are three 3-cycles; so altogether there are 12 even permutations:

$$\begin{gathered}\mathrm{Id},\ (1,2)(3,4),\ (1,3)(2,4),\ (1,4)(2,3),\\ (1,2,3),\ (1,3,2),\ (1,2,4),\ (1,4,2)\\ (1,3,4),\ (1,4,3),\ (2,3,4),\ (2,4,3).\end{gathered}$$

(These 12 permutations are the elements of the alternating group Alt(4).)

4. A certain permutation f, written as a product of disjoint cycles, has three 2-cycles, one 4-cycle, two 7-cycles, one 9-cycle and no other cycles (including no 1-cycles). To which symmetric group Sym(n) does f belong? Does f belong to Alt(n)? What is the order of f?

Solution.

Adding up the lengths of the cycles shows that f must be in Sym(33). There are 4 even length cycles; so f is even. So f belongs to Alt(33). The order of f is the smallest number that is divisible by 2, 4, 7 and 9. This is $4\times7\times9 = 252$.

5. In Sym(11), how many 3-cycles are there? (A 3-cycle must permute three of the numbers in a cycle and fix the remaining eight numbers.) How many 5-cycles are there? And how many permutations are there that are made up of a 2-cycle, a 4-cycle and a 5-cycle?

Solution.

There are $\binom{11}{3} = 165$ three-element subsets of the set of integers from 1 to 11. For each such set there are two ways of forming a cycle. So there are 330 3-cycles.

The number of five-element subsets is $\binom{11}{5} = 462$. Each five-element set gives 24 possible 5-cycles. To see this, choose any one of the five numbers, and consider what could follow it in the cycle. There are four possibilities. Once this has been chosen, there are three possibilities for the next thing in the cycle, and after that just two choices for what follows it. By then everything is settled. So there are $4 \times 3 \times 2 = 24$ possibilities. So the total number of 5-cycles is $462 \times 24 = 1108$.

The number of ways of splitting a set of 11 elements into subsets of sizes 2, 4 and 5 is $\binom{11}{2}\binom{9}{4}$. (Choose a 2-element subset first, and then choose a 4-element subset from the 9-element set that is left.) There is only one way to make a 2-cycle from two given elements, but four elements can be arranged into a 4-cycle in 6 ways, and five elements into a 5-cycle in 24 ways. So the total number of possibilities is

$$\binom{11}{2} \times \binom{9}{4} \times 6 \times 24 = \frac{11!}{2! \times 4! \times 5!} \times 6 \times 24 = \frac{11!}{40} = 997920.$$

6. Draw diagrams representing the permutations $(1,3)(2,4,5)$ and $(1,5)(2,4)$, and hence express them as products of transpositions that interchange adjacent numbers.

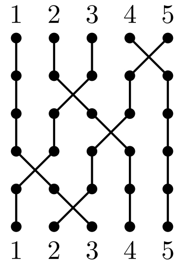
Solution.

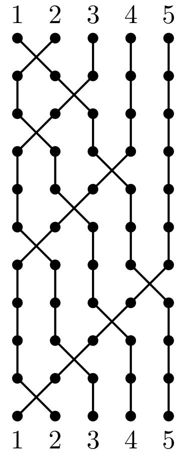
The diagrams below show that

$$(1,3)(2,4,5) = (4,5)(2,3)(3,4)(1,2)(2,3),$$

$$(1,5)(2,4) = (1,2)(2,3)(1,2)(3,4)(2,3)(1,2)(4,5)(3,4)(2,3)(1,2).$$

(There are many other solutions.)





7. What is the greatest possible order for an element of Sym(11)?

Solution.

You can determine the order of an element of Sym(n) from its "cycle type"—the number of cycles of each length when the permutation is written as a product of disjoint cycles. The order of the permutation is, in fact, the smallest number that is divisible by the lengths of all the cycles of the permutation.

To determine the maximum order for elements of Sym(11) you really have to consider all possible cycle types for elements of Sym(11). If the permutation is an 11-cycle then its order is 11. If it is a 10-cycle then its order is 10. If the permutation has a 9-cycle then it could be just a 9-cycle or the product of a 9-cycle and a 2-cycle. These possibilities give orders 9 and 18. If there is an 8-cycle then the permutation has order 8 (if it is just an 8-cycle or the product of an 8-cycle and a 2-cycle) or 24 (if it is the product of an 8-cycle and a 3-cycle). If there is a 7-cycle then the largest order you can get is 28, when the permutation is a 7-cycle by a 4-cycle, the other possibilities being 21 (7-cycle by 3-cycle) and 14 (7-cycle by 2-cycle or 7-cycle by 2-cycle by 2-cycle). If there is a 6-cycle then the largest possible order is 30, when the permutation is a 6-cycle by a 5-cycle, since if there were no 5-cycle the cycle lengths would all be divisors of 24. Suppose that the longest cycle is a 5-cycle. There could be two 5-cycles, but then the order would just be 5. If there is a 4-cycle with the 5-cycle then the order will be 20, whether the permutation is just a 5-cycle by a 4-cycle, or 5-cycle by 4-cycle by 2-cycle. If there is no cycle of length greater than 3 to go with the 5-cycle then the cycle lengths will all be divisors of 30—and 30 can occur, with 5-cycle by 3-cycle by 2-cycle. If there is no cycle of length greater than 4 then the cycle lengths will all be divisors of 24, and so the order cannot be more than 24.

The conclusion is that 30 is the largest possible order; it occurs for two different cycle types.