THE UNIVERSITY OF SYDNEY
MATH3906 Representation Theory
(http://www.maths.usyd.edu.au/u/UG/IM/MATH3906/)

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Tutorial 1

1. If A is an $m \times n$ matrix over the field $\mathbb{C}$ (complex numbers) then $\overline{A}$ is the $m \times n$ matrix whose entries are the complex conjugates of the entries of A.

(i) Show that an arbitrary complex matrix can be written as $P + iQ$ with P and Q real.

(ii) A is *Hermitian* if $\overline{A} = A^{\mathrm{t}}$ (transpose of A). Show that A is Hermitian if and only if its real part (P) is symmetric and its imaginary part (Q) is skew-symmetric.

(iii) Show that if A is Hermitian then $\overline{v}^{\mathrm{t}} A v$ is a real number for all complex column vectors v (of appropriate size).

(iv) A Hermitian matrix A is said to be it positive definite if $\overline{v}^{\mathrm{t}} A v > 0$ for all nonzero v. Prove that positive definite matrices are nonsingular.

(v) Show that a Hermitian matrix A is positive definite if and only if there exists a nonsingular B such that $A = \overline{B}^{\mathrm{t}} B$. (The "if" part is OK. For the "only if" you have to use row and column operations. Start by showing that the diagonal entries of A are real and positive.)

(vi) Show that the sum of two positive definite matrices is positive definite.

(vii) Let G be a finite subgroup of $\mathrm{GL}_n(\mathbb{C})$. Prove that there exists a positive definite matrix A such that $\overline{Y}^{\mathrm{t}} A Y = A$ for all $Y \in G$. (Hint: Try $A = \sum_{X \in G} \overline{X}^{\mathrm{t}} X$.)

(viii) Prove that if G is a finite subgroup of $\mathrm{GL}_n(\mathbb{C})$ then there exists a nonsingular B such that BXB^{-1} is unitary for all $X \in G$. (A matrix is unitary if its inverse is the transpose of its conjugate.)

Solution.

(i) Let A have (r,s)-entry $\alpha_{rs} \in \mathbb{C}$. Writing $\alpha_{rs} = \beta_{rs} + i\gamma_{rs}$ with $\beta_{rs}, \gamma_{rs} \in \mathbb{R}$ we see that $A = P + iQ$ where P and Q have (r,s)-entries β_{rs} and γ_{rs} (respectively).

(ii) Since $(\overline{P + iQ})^{\mathrm{t}} = (P - iQ)^{\mathrm{t}} = P^{\mathrm{t}} - iQ^{\mathrm{t}}$ we see that $A = P + iQ$ is Hermitian if and only if $P^{\mathrm{t}} = P$ and $Q^{\mathrm{t}} = -Q$.

(iii) Recall that transposing reverses products; that is, $(XY)^{\mathrm{t}} = Y^{\mathrm{t}} X^{\mathrm{t}}$ whenever the left hand side is defined. (Note that this implies that $(A^{-1})^{\mathrm{t}} = (A^{\mathrm{t}})^{-1}$ whenever A is nonsingular. It is also clear that taking complex conjugates preserves sums and products, and commutes with the maps $A \mapsto A^{-1}$ and $A \mapsto A^{\mathrm{t}}$.) Let v be an arbitrary column vector and let $z = \overline{v}^{\mathrm{t}} A v$. Since z is a 1×1 matrix we have $\overline{z}^{\mathrm{t}} = \overline{z}$, and so

$$\overline{z} = \left(\overline{(\overline{v}^{\mathrm{t}} A v)}\right)^{\mathrm{t}} = \overline{v}^{\mathrm{t}} \overline{A}^{\mathrm{t}} v = \overline{v}^{\mathrm{t}} A v = z.$$

Thus z is real.

(iv) Suppose that A is positive definite. Note first that since A is Hermitian it must be square (as its transpose is the same shape as itself). Now let v be in the nullspace of A; that is, v is a column vector such that $Av = 0$. Then $\overline{v}^{\mathrm{t}} A v = \overline{v} 0 = 0$, and positive definiteness of A gives $v = 0$. So the nullspace of A is $\{0\}$; this implies that A is nonsingular.

(v) Recall that if $v \in \mathbb{C}^n$ and the k^{th} entry of v is $x_k + iy_k$ (for $k = 1, 2, \ldots, n$) then $\overline{v}^{\mathrm{t}} v = \sum_{k=1}^{n} x_k^2 + y_k^2$, which is real, nonnegative, and zero only if $v = 0$. This shows that the identity matrix is positive definite. Suppose now that $A = \overline{B}^{\mathrm{t}} C B$ where $B, C \in \mathrm{GL}_n(\mathbb{C})$ and C is positive definite, and let $v \in \mathbb{C}^n$ be nonzero. Then $Bv \neq 0$, since B is nonsingular, and since C is positive definite it follows that $(\overline{Bv})^{\mathrm{t}} C (Bv) > 0$. But $\overline{v}^{\mathrm{t}} A v = (\overline{Bv})^{\mathrm{t}} C (Bv)$, and since this is positive for all nonzero v it follows that A is positive definite. Putting $C = I$ gives the "if" part.

Let A be an arbitrary positive definite $n \times n$ Hermitian matrix. We use induction on n to prove that A has the desired form; note that in the case $n = 1$ the matrix A is simply a positive real number, and we may take $B = \sqrt{A}$. Let e_l be the l^{th} column of the identity matrix (so that $e_1, e_2, \ldots, e_n$ comprise the standard basis of $\mathbb{C}^n$). The (l,l)-entry of A is $\overline{e_l}^{\mathrm{t}} A e_l$, which must be positive since A is positive definite. Thus we can write

$$A = \begin{pmatrix} a & \overline{\underset{\sim}{x}}^{\mathrm{t}} \\ \underset{\sim}{x} & A' \end{pmatrix}$$

where a is real and positive, $\underset{\sim}{x} \in \mathbb{C}^{n-1}$ and A' is some $(n-1) \times (n-1)$ Hermitian matrix. Now set

$$D = \begin{pmatrix} \sqrt{a^{-1}} & 0 \\ -a^{-1}\underset{\sim}{x} & I \end{pmatrix}$$

and observe that D is nonsingular; indeed, as a row operation matrix the effect of D is to divide the first row by $\sqrt{a}$ and add multiples of the first row to the others. We see that the first column of DA is $(\sqrt{a})e_1$. Now postmultiplication by $\overline{D}^{\mathrm{t}}$ performs a corresponding sequence of column operations, and we find that

$$DA\overline{D}^{\mathrm{t}} = \begin{pmatrix} 1 & 0 \\ 0 & A'' \end{pmatrix}.$$

Since A was positive definite, this must be too. Hence A'' is a $(n-1)\times(n-1)$ positive definite matrix. By induction we can write $A'' = \overline{Y}^{\mathrm{t}}Y$, and this gives

$$A = D^{-1}\begin{pmatrix}1 & 0\\ 0 & Y\end{pmatrix}^{\mathrm{t}}\begin{pmatrix}1 & 0\\ 0 & Y\end{pmatrix}(\overline{D}^{\mathrm{t}})^{-1} = \overline{B}^{\mathrm{t}}B$$

where $B = \left(\begin{smallmatrix}1 & 0\\ 0 & Y\end{smallmatrix}\right)(\overline{D}^{\mathrm{t}})^{-1}$.

(vi) If A, $B \in \mathrm{GL}_n(\mathbb{C})$ are positive definite and $0 \neq v \in \mathbb{C}^n$ then

$$\overline{v}^{\mathrm{t}}(A+B)v = \overline{v}^{\mathrm{t}}Av + \overline{v}^{\mathrm{t}}Bv > 0$$

(since $\overline{v}^{\mathrm{t}}Av > 0$ and $\overline{v}^{\mathrm{t}}Bv > 0$).

(vii) We see that $\overline{Y}^{\mathrm{t}}AY = \sum_{X\in G}(\overline{Y}^{\mathrm{t}}\overline{X}^{\mathrm{t}})(XY) = \sum_{Z\in G}\overline{Z}^{\mathrm{t}}Z = A$ (since $Z = XY$ runs through all elements of G as X does).

(viii) By *(vii)* we can find a positive definite A such that $\overline{Y}^{\mathrm{t}}AY = A$ for all $Y \in G$, and by (v) we can put $A = \overline{B}^{\mathrm{t}}B$. But the equation $\overline{Y}^{\mathrm{t}}\overline{B}^{\mathrm{t}}BY = \overline{B}^{\mathrm{t}}B$ can be written as $BY^{-1}B^{-1} = (\overline{B}^{-1})^{\mathrm{t}}\overline{Y}^{\mathrm{t}}\overline{B}^{\mathrm{t}}$, or, equivalently,

$$(BYB^{-1})^{-1} = (\overline{BYB^{-1}})^{\mathrm{t}},$$

showing that BYB^{-1} is unitary for all $Y \in G$.

2. Recall that the dot product on $\mathbb{C}^n$ is defined by $u\cdot v = \overline{u}^{\mathrm{t}}v$, and that unitary matrices preserve it (in the sense that $(Xu)\cdot(Xv) = u\cdot v$ for all u and v if X is unitary). Recall also that if U is a subspace of $\mathbb{C}^n$ then $\mathbb{C}^n = U \oplus U^{\perp}$, where

$$U^{\perp} = \{\, v\in\mathbb{C}^n \mid u\cdot v = 0 \text{ for all } u\in U\,\}$$

(the orthogonal complement of U).

Let G be a finite group of $n\times n$ unitary matrices, and let U be a G-invariant subspace of $\mathbb{C}^n$. (That is, if $X\in G$ and $u\in U$ then $Xu\in U$.) Prove that the orthogonal complement of U is also G-invariant.

Solution.

Let $v\in U^{\perp}$ and let $X\in G$. Then for all $u\in U$ we have that $X^{-1}u\in U$ (since $X^{-1}\in G$ and U is G-invariant), and so

$$\begin{aligned}(Xv)\cdot u &= Xv\cdot X(X^{-1}u)\\ &= v\cdot X^{-1}u \qquad \text{(since } X \text{ is unitary)}\\ &= 0 \qquad \text{(since } v\in U^{\perp}).\end{aligned}$$

Hence $Xv\in U^{\perp}$, and since this holds for all $X\in G$ and $v\in U^{\perp}$ we have shown that $U^{\perp}$ is G-invariant.

3. Let H and N be groups and $\phi\colon H\to\mathrm{Aut}(N)$ a homomorphism. Define

$$H\ltimes N = \{\,(h,x)\mid h\in H,\ x\in N\,\}$$

with multiplication given by

$$(h,x)(k,y) = (hk, x^{\phi(k)}y)$$

for all $h,k\in H$ and $x,y\in N$. Prove that this makes $H\ltimes N$ into a group. (Such a group is called a *semidirect product* of N by H. If ϕ is the trivial homomorphism ($h\mapsto 1\in\mathrm{Aut}(N)$ for all $h\in H$) we get the direct product of N and H.)

Solution.

Since ϕ is a homomorphism we have $\phi(1)=1$, where the 1 on the left hand side is the identity element of H and the 1 on the right hand side is the identity automorphism of N. Hence our multiplication rule gives

$$(h,x)(1,1) = (h1, x^{\phi(1)}1 = (h,x).$$

Since all automorphisms of N map 1 to 1 we also find that

$$(1,1)(h,x) = (1h, 1^{\phi(h)}x) = (h,x).$$

So $H\ltimes N$ has an identity element. The following calculation proves associativity:

$$\begin{aligned}\big((h,x)(k,y)\big)(l,z) &= (hk, x^{\phi(k)}y)(l,z) = (hkl, (x^{\phi(k)}y)^{\phi(l)}z)\\ (hkl, x^{\phi(k)\phi(l)}y^{\phi(l)}z) &= (hkl, x^{\phi(kl)}y^{\phi(l)}z)\\ &= (h,x)(kl, y^{\phi(l)}z) = (h,x)\big((k,y)(l,z)\big).\end{aligned}$$

Let (h,x) be an arbitrary element of $H\ltimes N$, and let $k=h^{-1}$ and $y=(x^{-1})^{\phi(k)}$. Since $(x^{-1})^{\phi(k)} = (x^{\phi(k)})^{-1}$ we see that $(h,x)(k,y) = (hk, x^{\phi(k)}y) = (1,1)$. Moreover, since $\phi(k)\phi(h) = \phi(hk) = 1$ we also have that $y^{\phi(h)} = x^{-1}$, and $(k,y)(h,x) = (kh, y^{\phi(h)}x = (1,1)$, so that (k,y) is definitely the inverse of (h,x).