Ramsey-type colourings and Relation Algebras

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We want to colour the edges of K_n with two colours, so that there be no monochromatic triangles. What is the largest n for which K_n admits such a colouring?

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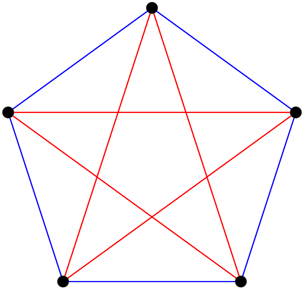

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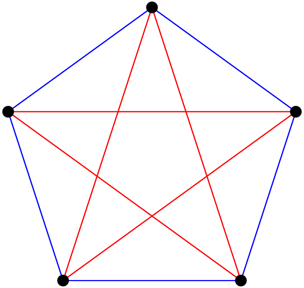

This answer is also the smallest possible if we want to satisfy the following principle: Every triangle that is not forbidden, occurs everywhere it can.

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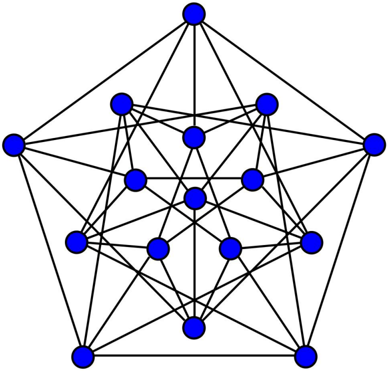
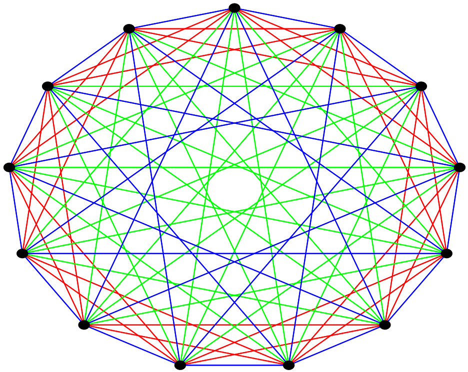


Three colours

On K_{13}. It can be shown that nothing smaller will do.



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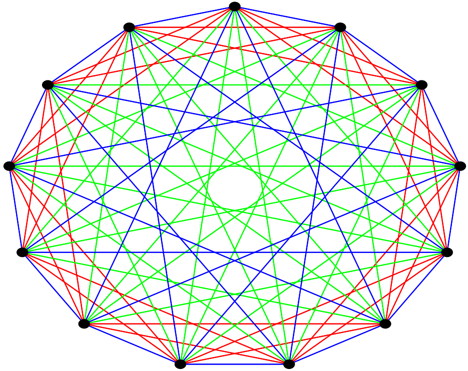

Interestingly, neither will K_{14} and K_{15}.

In terms of Ramsey numbers

Let $\langle n_1, \ldots, n_k \rangle$ be a finite sequence of natural numbers. Consider k-colourings of the edges of a K_m, such that for every $i \leq k$ there are no n_i-cliques.

Theorem (essentially Ramsey, 1928)

For any $\langle n_1, \ldots, n_k \rangle$ there is a finite bound on m for which such a colouring of K_m exists.

Let $R(n_1, \ldots, n_k)$ stand for the smallest m for which the required colouring does not exist.

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- $R(3,3,3,3) \leq 66$, $R(3,3,3,3,3) \leq 327$, ...
- In general $R(3^{n+1}) \leq (n+1)(R(3^n) - 1) + 2$.

In terms of relation algebras

A Ramsey Relation Algebra (RaRA) $\mathbf{M}_n$ is a finite relation algebra on $n+1$ atoms $1', a_1, \ldots, a_n$, whose composition table is given by:

$$1' ; a_i = a_i = a_i ; 1' \quad \text{and} \quad a_i ; a_j = \begin{cases} 0' & \text{if } i \neq j \\ a_i^- & \text{if } i = j \end{cases}$$

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- For $n = 4, 5$ representations were constructed by S. Comer.
- For $n > 5$ the representability question was open.

A closer look at the pentagon

Consider $\mathbb{Z}_5$ as a finite field. Let g be a generator of its multiplicative group $\mathbb{Z}_5^*$. Order of $\mathbb{Z}_5^*$ happens to be divisible by the number of colours, so we build a rectangular matrix

$$\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} \cong \begin{pmatrix} g & g^3 \\ g^2 & g^4 \end{pmatrix} \cong \begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix}$$

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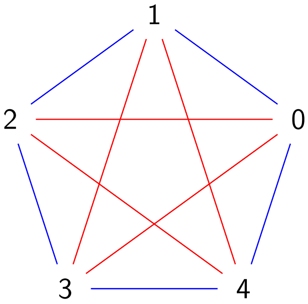
And this is what we get:



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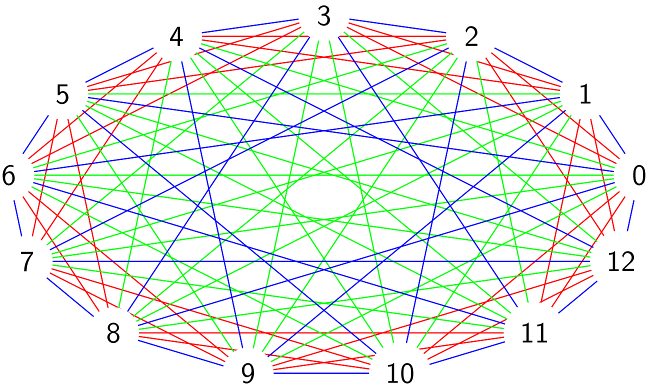

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For n colours

If it works for 2 and 3, then it must work for any n, must it not? Suppose we have found $GF(p^K)$, such that n divides $p^K - 1$. Put $(p^K - 1)/n = m$. Let g be a generator of the multiplicative group of $GF(p^K)$, and M be the $n \times m$ matrix

$$\begin{pmatrix} g & g^{n+1} & \dots & g^{(m-1)n+1} \\ g^2 & g^{n+2} & & g^{(m-1)n+2} \\ \vdots & \vdots & & \vdots \\ g^n & g^{n+n} & \dots & g^{(m-1)n+n} \end{pmatrix}$$

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where $g^{(m-1)n+n} = g^{mn} = 1$. We will write R_i for the i–th row of M, considered as a set. The complex operations on the rows have their usual meaning, that is

$$-R_i = \{-g^i, -g^{n+i}, \dots, -g^{(m-1)n+i}\}$$

and

$$R_i + R_j = \{a + b \colon a \in R_i,\ b \in R_j\}.$$

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Lemma

If (i)–(iv) above hold, then:

1. $-R_i = R_i$,
2. $R_i + R_i = \bigcup_{j \neq i} R_j$,
3. $R_i + R_j = M$, *if* $i \neq j$,

for every $i, j \in \{1, \dots, m\}$.

Representations (colourings)

Theorem

Let $\mathbf{M}_n$ be a Ramsey algebra, and $GF(p^K)$ is such that n divides $p^K - 1$. Put $m = (p^K - 1)/n$ and let M be an $n \times m$ matrix over $GF(p^K)$ constructed as before. Suppose M satisfies the representability conditions (i)–(iv). Then

- *$\mathbf{M}_n$ is representable over $GF(p^K)$ — more precisely, over the additive group of $GF(p^K)$.*
- *The representation of $\mathbf{M}_n$ is the subalgebra of the complex algebra of the additive group of $GF(p^K)$, whose atoms are the sets $\{0\}$ and R_i, for $i \in \{1, \ldots, n\}$.*

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That is, if for a given n we can find a suitable finite field, then all is well. But can we?

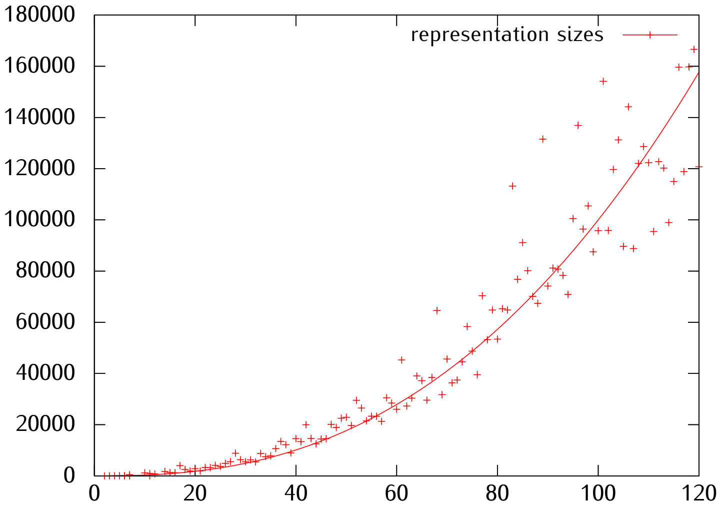
Three oddities

Clrs	Repres. over p^K	Upper bound	Comment
2	5	5	unique
3	$13,\ 2^4 = 16$	16	Ramsey bound attained
4	41	65	exhaustive
5	71, 101	326	exhaustive
6	97, 157, 277	1957	exhaustive
7	491	13700	exhaustive
8	none	109601	exhaustive
9	$19^2 = 361$	986410	exh., no prime field repr.
10	1181	9864101	exhaustive
11	947, 1409	108505112	not exhaustive
12	769, 1201	1032061345	not exhaustive
13	???	13416797486	not exhaustive

Colourings (representations) over prime fields

n	repr.	*n*	repr.	*n*	repr.	*n*	repr.	*n*	repr.
		25	3701	49	22541	73	44531	97	96419
2	5	26	4889	50	22901	74	58313	98	105449
3	13	27	5563	51	19687	75	48751	99	87517
4	41	28	8849	52	29537	76	39521	100	95801
5	71	29	6323	53	26501	77	70379	101	154127
6	97	30	5521	54	21493	78	53197	102	95881
7	491	31	6263	55	23321	79	64781	103	119687
8		32	5441	56	23297	80	53441	104	131249
9		33	8779	57	21319	81	65287	105	89671
10	1181	34	7481	58	30509	82	64781	106	144161
11	947	35	7841	59	28439	83	113213	107	88811
12	769	36	10657	60	26041	84	76777	108	122041
13		37	13469	61	45263	85	91121	109	128621
14	1709	38	12161	62	27281	86	80153	110	122321
15	1291	39	8971	63	30367	87	70123	111	95461
16	1217	40	14561	64	39041	88	67409	112	122753
17	4013	41	13367	65	37181	89	131543	113	120233
18	2521	42	19993	66	29569	90	74161	114	98953
19	1901	43	14621	67	38459	91	81173	115	115001
20	2801	44	12497	68	64601	92	80777	116	159617
21	1933	45	14401	69	31741	93	78307	117	118873
22	3257	46	14537	70	45641	94	70877	118	159773
23	3221	47	20117	71	36353	95	100511	119	166601
24	4129	48	18913	72	37441	96	136897	120	120721

We are doing science



And (very little) maths

Conjecture

Let $n > 13$. Then there exist a prime p such that n divides $p-1$ and the $n \times m$ matrix (with $m = (p-1)/n$)

$$\begin{pmatrix} g & g^{n+1} & \dots & g^{(m-1)n+1} \\ g^2 & g^{n+2} & & g^{(m-1)n+2} \\ \vdots & \vdots & & \vdots \\ g^n & g^{n+n} & \dots & g^{(m-1)n+n} \end{pmatrix}$$

over $GF(p)$ satisfies representability conditions for n.