Non-Archimedean Geometry

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Plan

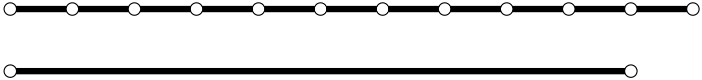
Introduction to non-Archimedean geometry.

Berkovich spaces.

Some recent progress.

The Archimedean Axiom

The Archimedean axiom states that sufficiently many copies of a given line segments together become longer than any other given line segment.



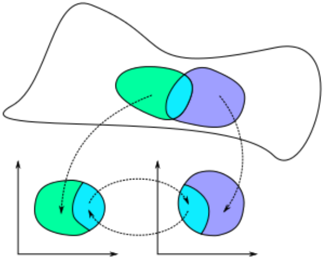
Attributed to Archimedes of Syracuse (287-212 BC). Archimedes himself attributed it to Eudoxus of Cnidus (408-355 BC).

Roughly speaking, non-Archimedean geometry is geometry without this axiom. In practice, this can mean different things.

We shall view non-Archimedean geometry as an analogue of complex geometry.

Differential geometry

The main objects in differential geometry are manifolds. These locally look like Euclidean space $\mathbf{R}^n$.



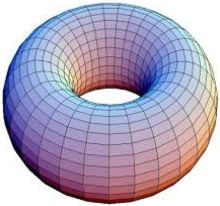
Transition functions are smooth (differentiable functions).

Manifolds can have interesting topology.

Manifolds come with differential forms and other objects.

Can add/study additional structures: line bundles, vector bundles, and connections and metrics on these.

Complex geometry



Complex manifolds locally look like balls in $\mathbf{C}^n$.

Transition functions are analytic: locally given by convergent power series.

Complex manifolds are more rigid than differentiable manifolds. One reason is uniqueness of analytic continuation: two analytic functions that agree on an open set must be the same.

Example: the zero set $\bigcap_i \{f_i = 0\}$ of a set of polynomials f_i with complex coefficients is often a complex manifold.

Valued fields

Now replace $\mathbf{C}$ another *valued field* i.e. a field K with a norm $|\cdot| : K \to [0, \infty)$ satisfying

$$|a| = 0 \iff a = 0,$$
$$|a + b| \leq |a| + |b| \quad \text{and} \quad |ab| = |a||b|.$$

Examples:

- $\mathbf{C}$ or $\mathbf{R}$ with the usual norm.
- $\mathbf{Q}$ with the p-adic norm: $|p| < 1$.
- Formal Laurent series $\mathbf{C}((t))$: $|t| < 1$.
- Any field K with the *trivial norm*: $|a| = 1$ if $a \neq 0$.

Can talk about convergent power series with coefficients in K.

Can we do analytic geometry over general valued fields?

Non-Archimedean fields

The valued fields

- $\mathbf{Q}$ with the p-adic norm
- Formal Laurent series $\mathbf{C}((t))$
- Any field K with the *trivial norm*

are *non-Archimedean*: they satisfy the strong triangle inequality

$$|a+b| \leq \max\{|a|, |b|\}.$$

Relation to the Archimedean axiom: K is not non-Archimedean iff there exists an integer $n > 1$ such that

$$|n| = |1 + 1 + \cdots + 1| > 1.$$

Fact: most valued fields are non-Archimedean!

Can we do analytic geometry over general valued fields?

Why non-Archimedean Geometry

General curiosity!

Number theory: study complexity of numbers by looking at their size under different absolute values. Based on...

Ostrowski's Theorem: any absolute value on $\mathbf{Q}$ is equivalent to one of the following:

- The trivial norm.
- The usual Archimedean norm.
- The p-adic norm for some prime p.

Degenerations. Can use $\mathbf{C}((t))$ to describe degenerations of complex manifolds.

Can also use $\mathbf{C}$ with the trivial norm to describe degenerations.

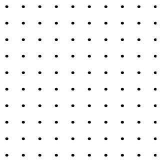
Naive approach

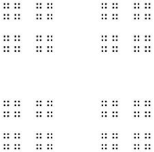
Can try to develop analytic geometry as with complex manifolds: glueing together balls using analytic functions as "glue".

Problem 1: balls in K^n are totally disconnected and look like:

A discrete set if K is trivially valued.

A Cantor set otherwise.





Glueing together such sets won't yield interesting topology!

Problem 2: uniqueness of analytic continuation fails.

Too many analytic functions with the naive definition. (This is related to Problem 1).

Berkovich spaces

There have been several approaches to developing analytic geometry over non-Archimedean fields:

- M. Krasner (1940's): function theory.
- J. Tate (1960's): rigid spaces.
- V. Berkovich (1980's-1990's): analytic spaces (now known as Berkovich spaces).
- R. Huber (1990's): adic spaces.

I will focus on Berkovich spaces.

Berkovich spaces are obtained by gluing together basic building blocks, called *affinoids*.

Definition of affinoids and (especially) gluing is a bit tricky...

...but it's easy to define what the Berkovich affine space is!

The Berkovich affine space

The Berkovich affine n-space $\mathbf{A}_K^{n,\mathrm{an}}$ over K is the set of all multiplicative seminorms

$$|\cdot| : K[z_1, \ldots, z_n] \to [0, \infty)$$

that extend the given norm on K.

Every point $a \in K^n$ defines a seminorm in $\mathbf{A}_K^{n,\mathrm{an}}$:

$$f \mapsto |f(a)|$$

so we get an injective map $K^n \hookrightarrow \mathbf{A}_K^{n,\mathrm{an}}$.

If $K = \mathbf{C}$ with the usual norm, Gelfand-Mazur implies that this map is surjective, so $\mathbf{A}_{\mathbf{C}}^{n,\mathrm{an}} = \mathbf{C}^n$.

However, if K is non-Archimedean, $\mathbf{A}_K^{n,\mathrm{an}} \supsetneq K^n$.

Thm. $\mathbf{A}_K^{n,\mathrm{an}}$ is locally compact and path connected.

Can similarly define the analytification X^{an} of a general algebraic variety (or scheme) X over K.

The Berkovich affine line

Can describe the Berkovich affine line $\mathbf{A}_K^{1,\mathrm{an}}$ quite concretely.

Exemplify for a *trivially valued and algebraically closed* field K.

$\mathbf{A}_K^{1,\mathrm{an}}$ is the set of seminorms $|\cdot|$ on $K[z]$ that are trivial on K.

Case 1: $|\cdot|$ is trivial on K.

Case 2: $|z| > 1$. The seminorm is then uniquely determined by $r := |z| \in (1, \infty)$:

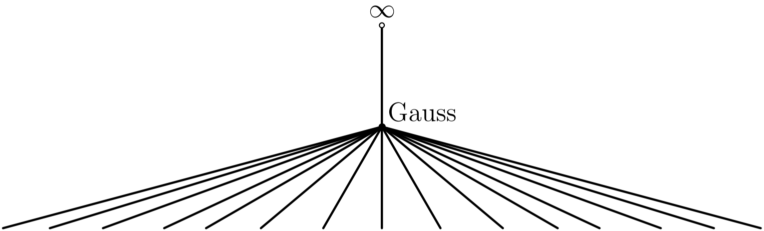
$$|\sum_j c_j z^j| = \max_j |c_j z^j| = \max_{c_j \neq 0} r^j$$

Case 3: $\exists! a \in K$ such that $|z - a| < 1$. The seminorm is then uniquely determined by a and by $r := |z - a| \in [0, 1)$:

$$|\sum_j c_j (z-a)^j| = \max_j |c_j (z-a)^j| = \max_{c_j \neq 0} r^j$$

The Berkovich affine line

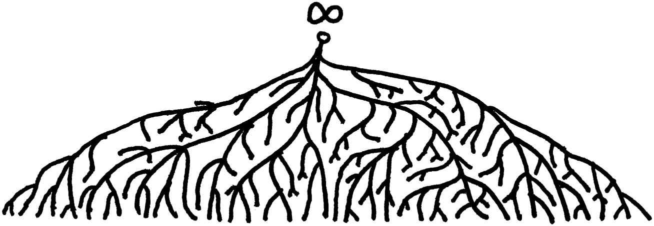
$\mathbf{A}_K^{1,\mathrm{an}}$ looks like a tree with endpoints given by $K \cup \{\infty\}$ and a simple branch point.



The branch point is the trivial norm (called the Gauss point).

The Berkovich affine line

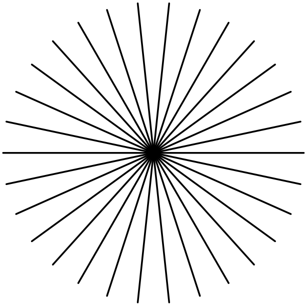
When K is non-Archimedean and non-trivially valued, the Berkovich affine line looks like a tree with lots of branching

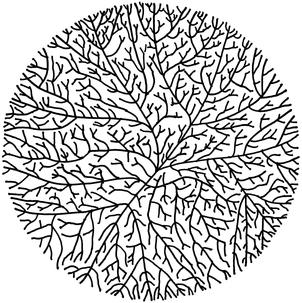


Every point in $K \cup \{\infty\}$ determines an endpoint in $\mathbf{A}_K^{1,\mathrm{an}}$ but there may be other endpoints, too (depends on K).

The Berkovich projective line and other curves

Get the Berkovich projective line $\mathbf{P}^{1,\mathrm{an}}_K = \mathbf{A}^{1,\mathrm{an}}_K \cup \{\infty\}$ by adding one point. This is also a tree.





Berkovich curves of higher genus look like trees except that they may have finitely many loops. (Matt Baker calls them *arboreta*).

Higher-dimensional Berkovich spaces

Higher dimensional Berkovich spaces are harder to visualize, but:

- The Berkovish affine plane over a trivially valued field "locally" looks like a cone over an **R**-tree.
- Many Berkovich spaces can be viewed as an inverse limit of simplicial complexes (Kontsevich-Soibelman, Payne, Foster-Gross-Payne, Boucksom-Favre-J).

In any case, it is not always necessary to visualize a Berkovich space in order to work with it!

Recent work/progress on Berkovich spaces

- Topological structure of Berkovich spaces (Berkovich, Hrushkovski-Loeser, Thuillier).
- Differential forms (and currents) on Berkovich spaces. (Chambert-Loir and Ducros).
- Dynamics (Rivera, Favre, J, Kiwi, Baker, Rumely, Yuan, Zhang, Ruggiero, Gignac, DeMarco, Faber,. . .)
- Connections to tropical geometry [Payne, Baker-Payne-Rabinowitz, Foster-Gross-Payne,. . .]
- Differential Equations [Kedlaya, Poineau, Pulita. . .]
- Potential Theory [Zhang, Kontsevich-Tschinkel, Thuillier, Baker-Rumely, Boucksom-Favre-J. . .]
- Metrics on line bundles on Berkovich spaces (Zhang, Gubler, Chambert-Loir, Ducros, Yuan, Liu, Boucksom-Favre-J).
- etc. . .

Two results

Now describe two types of results in non-Archimedean geometry and dynamics.

(1) A non-Archimedean Calabi-Yau theorem.

(2) Degree growth of plane polynomial maps.

The result in (1) is *formulated* in terms of Berkovich spaces (but may be useful in the study of Kähler metrics).

In (2) the formulation of the problem does not involve Berkovich spaces, but the *method* does.

Complex Calabi-Yau Theorem

$X =$ smooth, complex projective variety

$L \to X$ ample line bundle.

$c_1(L, \|\cdot\|)$ curvature form of smooth metric $\|\cdot\|$ on L.

Calabi-Yau Theorem: if μ is a smooth positive measure on X of mass $c_1(L)^n$, then there exists a unique (up to scaling) positive metric $\|\cdot\|$ on L such that

$$c_1(L, \|\cdot\|)^n = \mu. \tag{1}$$

Can view (1) as nonlinear PDE, a *Monge-Ampère equation*.

Kołodziej, and later Guedj and Zeriahi, generalized this to the case when μ is a positive, not too singular measure. Metric is then no longer smooth.

A non-Archimedean Calabi-Yau theorem

$K =$ discretely valued field of residue characteristic 0, or trivially valued field of characteristic 0.

$X =$ smooth projective variety over K.

$L \to X$ ample line bundle.

$\mu =$ not too singular measure on Berkovich space X^{an}.

Thm [Boucksom-Favre-J, 2012,2013] There exists a semipositive continuous metric $\|\cdot\|$ on L such that $c_1(L, \|\cdot\|)^n = \mu$. The metric is unique up to a multiplicative constant.

All concepts in theorem were previsouly introduced by Berkovich, Zhang, Gubler, Chambert-Loir...

Previous results:

Kontsevich-Tschinkel (2001): sketch for $\mu =$ Dirac mass.
Thuillier (2005): $X =$ curve.
Yuan-Zhang (2009): uniqueness.
Liu (2010): existence when $X =$ abelian variety.

Degree growth of plane polynomial maps

$f : \mathbf{C}^2 \to \mathbf{C}^2$ polynomial mapping.

$f(x,y) = (p(x,y), q(x,y))$.

$\deg f := \max\{\deg p, \deg q\}$

$f^n := f \circ f \circ \cdots \circ f$ (n times).

Easy to see that $\deg f^{n+m} \le \deg f^n \cdot f^m$.

Define (first) *dynamical degree* as

$$\lambda_1 := \lim_{n\to\infty} (\deg f^n)^{1/n} = \inf_{n\to\infty} (\deg f^n)^{1/n}$$

Called *algebraic entropy* by Bellon-Viallet.

Example: $f(x,y) = (y, xy)$

$f^2(x,y) = (xy, xy^2)$, $f^3(x,y) = (xy^2, x^2y^3), \ldots$

$\deg f^n = \mathrm{Fibonacci}_{n+2}$,

$\lambda_1 = \frac{\sqrt{5}+1}{2}$.

Degree growth of plane polynomial maps

Thm (Favre-J, 2011) For any polynomial map $f : \mathbf{C}^2 \to \mathbf{C}^2$ the following holds:

- The dynamical degree λ_1 is a quadratic integer.
- The sequence $(\deg f^n)_{n=1}^{\infty}$ satisfies integral linear recursion: there exists $k \geq 1$ and $c_j \in \mathbf{Z}$, $1 \leq j \leq k$, such that

$$\deg f^n = \sum_{j=1}^{k} c_j \deg f^{n-j}$$

Related work by Gignac and Ruggiero (2013).

Main idea is to study the induced action on the Berkovich affine plane $\mathbf{A}^{2,\mathrm{an}}_{\mathbf{C}}$ over the field $\mathbf{C}$ *equipped with the trivial norm*.

Exploit the fact that this affine plane has an invariant subset with the structure of a cone over a tree.

Unclear if the theorem is true in dimension ≥ 3.

Degree growth: outline of proof

Look at induced map on the Berkovich space $\mathbf{A}_{\mathbf{C}}^{2,\mathrm{an}}$.

$\mathbf{A}_{\mathbf{C}}^{2,\mathrm{an}}$ contains invariant subset which is a cone over an $\mathbf{R}$-tree.

Perron-Frobenius-type argument gives point $x \in \mathbf{A}_{\mathbf{C}}^{2,\mathrm{an}}$ such that

$$f(x) = \lambda_1 \cdot x$$

Several cases. In one case, the value group of x has rank two, generated by $\alpha_1, \alpha_2 \in \mathbf{R}_{>0}$, with $\alpha_1/\alpha_2 \notin \mathbf{Q}$.

The equation $f(x) = \lambda_1 \cdot x$ leads to

$$\lambda_1 \alpha_i \in \mathbf{Z}\alpha_1 + \mathbf{Z}\alpha_2, \quad i = 1, 2.$$

Thus λ_1 is a quadratic integer!

The proof that the sequence $(\deg f^n)_{n=1}^{\infty}$ satisfies linear recursion requires more work but uses that the ray $\mathbf{R}_{>0}$ is *attracting*.