Fundamental Groups of Flat Pseudo-Riemannian Spaces

Wolfgang Globke

School of Mathematical Sciences

THE UNIVERSITY of ADELAIDE

57th Annual Meeting of the Australian Mathematical Society

A flat manifold is a smooth manifold M with a torsion-free affine connection ∇ of curvature 0.

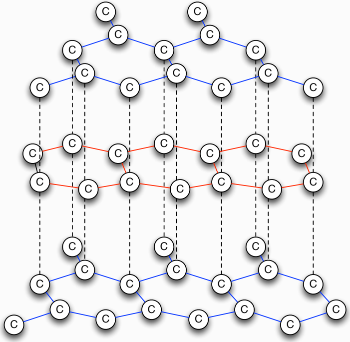
- M geodesically complete:
  - $M = \mathbb{R}^n/\Gamma$
  - $\Gamma \subset \mathbf{Aff}(\mathbb{R}^n)$
  - Γ-action free and properly discontinuous
- M not complete:
  - $M = \mathfrak{D}/\Gamma$
  - $\mathfrak{D} \subset \mathbb{R}^n$ open and Γ-invariant

I Classical Results on Crystallographic Groups

Tilings and Crystals

Photographs by John Baez

`http://math.ucr.edu/home/baez/alhambra/`



$\Gamma \subset \mathbf{Euc}(\mathbb{R}^n)$ is a crystallographic group if it is

- discrete (as a subset of $\mathbf{Euc}(\mathbb{R}^n)$),
- cocompact (has relatively compact fundamental domain).

If Γ is also torsion-free (no $\gamma \neq \mathrm{id}$ of finite order)
then Γ is called a Bieberbach group.

Hilbert's 18^{th} Problem

"Is there in n-dimensional Euclidean space [. . .] only a finite number of essentially different kinds of groups of motions with a [compact] fundamental region?"

Bieberbach Theorems (1911/1912)

Bieberbach I

Let Γ *be a crystallographic group. Then:*

- $\Gamma \cap \mathbb{R}^n$ *is a lattice in* $\mathbb{R}^n$.
- $\textsc{lin}(\Gamma)$ *is finite and faithfully represented in* $\mathbf{GL}_n(\mathbb{Z})$.

Bieberbach II

Let Γ_1, Γ_2 *be crystallographic groups. Then:*
$\Gamma_1 \cong \Gamma_2 \quad \Leftrightarrow \quad \Gamma_1$ *and* Γ_2 *affinely equivalent.*

Bieberbach III

For given dimension n, there exist only finitely many (affine equivalence classes of) crystallographic groups.

Let $M = \mathbb{R}^n/\Gamma$ be a compact complete connected flat Riemannian manifold.

The fundamental group $\Gamma \subset \mathbf{Euc}(\mathbb{R}^n)$ is ...

- discrete,
- torsion-free,
- cocompact.

In other words: Γ is a Bieberbach group.

Geometric Bieberbach Theorems

Bieberbach I*

Let M be a compact complete connected flat Riemannian manifold. Then:

- *M is finitely covered by a flat torus.*
- *$\mathbf{Hol}(M)$ is finite.*

Bieberbach II*

Let M_1 and M_2 be a compact complete connected flat Riemannian manifolds with fundamental groups Γ_1 and Γ_2. Then:

$\Gamma_1 \cong \Gamma_2 \quad \Leftrightarrow \quad M_1$ *and* M_2 *are affinely equivalent.*

Bieberbach III*

For a given dimension n, there are only finitely many equivalence classes of compact complete connected flat Riemannian manifolds.

n	crystallographic	Bieberbach
2	17	2
3	219 (or 230)	10
4	4783	74
5	222018	1060
6	28927915	38746

II Affine Crystallographic Groups

Riemannian manifold $\rightsquigarrow$ affine manifold:

- $\Gamma \subset \mathbf{Euc}(\mathbb{R}^n) \rightsquigarrow \Gamma \subset \mathbf{Aff}(\mathbb{R}^n)$.
- Γ discrete, torsion-free, cocompact
  $\rightsquigarrow$ Γ-action properly discontinuous, free, (cocompact).

Γ is an affine crystallographic group.

Do Bieberbach's Theorems hold in this setting?

No! Counterexamples to all three theorems exist.

Auslander Conjecture (1964)

A tentative analogue to Bieberbach's First Theorem:

Conjecture

If $\Gamma \subset \mathbf{Aff}(\mathbb{R}^n)$ is an affine crystallographic group,
then Γ is virtually polycyclic.

A group Γ is called...

- polycyclic if there exists a sequence of subgroups

$$\Gamma = \Gamma_0 \supset \Gamma_1 \supset \ldots \supset \Gamma_k = \mathbf{1}$$

  such that all Γ_j / Γ_{j+1} are cyclic groups.
- virtually polycyclic if Γ contains a polycyclic subgroup Γ' of finite index (also: polycyclic-by-finite).

Milnor Theorem and Conjecture (1977)

Theorem

Let Γ be a torsion-free and virtually polycyclic group. Then:
Γ is the fundamental group of some complete flat affine manifold.

Conjecture

The fundamental group of a flat affine manifold is virtually polycyclic.

Margulis (1983): Milnor's conjecture is wrong!
Non-abelian free $\Gamma \subset \mathbf{O}_{2,1} \ltimes \mathbb{R}^3$ exist.

Special Cases

Auslander's Conjecture has been proven in special cases:

- $\Gamma \subset \mathbf{Aff}(\mathbb{R}^3)$ (Fried & Goldman, 1983)
- $\Gamma \subset \mathbf{Iso}(\mathbb{R}^n_1)$ (Lorentz metric)
  - Conjecture holds for complete compact flat Lorentz manifolds (Goldman & Kamishima, 1984)
  - Compact flat Lorentz manifolds are complete (Carriere, 1989)
  - Classification is known (Grunewald & Margulis, 1989)

III Flat Pseudo-Riemannian Homogeneous Spaces

Let $M = \mathbb{R}^n/\Gamma$. Then:
M homogeneous $\quad\Leftrightarrow\quad \mathrm{Z}_{\mathbf{Aff}(\mathbb{R}^n)}(\Gamma)$ acts transitively on $\mathbb{R}^n$.

Theorem (Wolf, 1962)
Let Γ be the fundamental group of a flat pseudo-Riemannian homogeneous manifold M. Then:

- *Γ is 2-step nilpotent ($[\Gamma,[\Gamma,\Gamma]] = \mathbf{1}$).*
- *$\gamma = (I_n + A, v) \in \Gamma$ with $A^2 = 0$ and $Av = 0$ (unipotent).*
- *Γ abelian in signatures $(n,0)$, $(n-1,1)$, $(n-2,2)$.*

1. Is Γ always abelian?
2. If not, is $\textsc{lin}(\Gamma)$ ($= \mathbf{Hol}(M)$) always abelian?
3. Which Γ appear as fundamental groups of flat pseudo-Riemannian homogeneous spaces?
4. And what about the compact case?

Baues (2010):

- Example of non-abelian Γ with abelian $\textsc{lin}(\Gamma)$, $\dim M = 6$.
- Compact M always has abelian $\textsc{lin}(\Gamma)$.

Theorem (Globke, 2011)
Let M be a flat pseudo-Riemannian homogeneous manifold. If $\mathbf{Hol}(M)$ is not abelian, then

$$\dim M \geq 8.$$

If in addition M is complete, then

$$\dim M \geq 14.$$

Examples show that both bounds are sharp.

Theorem (Globke, 2012)

Let Γ be a group,

- *finitely generated*
- *torsion-free*
- *2-step nilpotent of rank n.*

Then:

Γ is the fundamental group of a complete flat pseudo-Riemannian homogeneous manifold M, and $\dim M = 2n$.

References

- O. Baues, Prehomogeneous Affine Representations and Flat Pseudo-Riemannian Manifolds, in ‘Handbook of Pseudo-Riemannian Geometry’, EMS, 2010
- L. Bieberbach, Über die Bewegungsgruppen der Euklidischen Räume, Math. Ann. 70, 1911 & Math. Ann. 72, 1912
- L.S. Charlap, Bieberbach Groups and Flat Manifolds, Springer, 1986
- J.H. Conway, H. Burgiel, C. Goodman-Strauss, The Symmetries of Things, A.K. Peters, 2008
- W. Globke, Holonomy Groups of Flat Pseudo-Riemannian Homogeneous Manifolds, Dissertation, Karlsruhe Institute of Technology, 2011
- D. Fried, W.M. Goldman, M.W. Hirsch, Affine manifolds with nilpotent holonomy, Comment. Math. Helvetici 56, 1981
- J. Milnor, Hilbert’s problem 18: On crystallographic groups, fundamental domains, and on sphere packings, in *John Milnor: Collected Papers I*, Publish or Perish, 1994
- J.A. Wolf, Spaces of Constant Curvature, 6^{th} ed., AMS Chelsea Publishing, 2011