ON SOME GEOMETRY OF DIFFERENTIAL EQUATIONS

AUSTRALIAN MATHEMATICAL SOCIETY, SYDNEY
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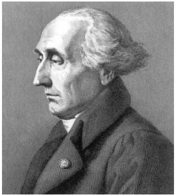
ROBERT L. BRYANT — DUKE UNIVERSITY

Euler   Lagrange

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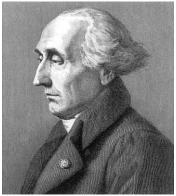
Euler   Lagrange

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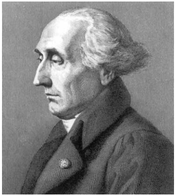
Euler   Lagrange

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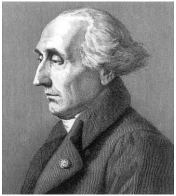
Euler   Lagrange

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Example: On a general surface in 3-space, the condition is that the acceleration vector of the curve at each point be perpendicular to the surface's tangent plane at that point. (E.g., great circles on spheres.)

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Gauss studied the properties of a surface S in space that depended only on the length of curves in the surface, i.e., on the element of arc

$$ds^2 = dx^2 + dy^2 + dz^2 = E(u,v)\,du^2 + 2F(u,v)\,du\,dv + G(u,v)\,dv^2$$

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He proved (*Theorem Egregium*) that $K = \kappa_1\kappa_2$ can be computed using only E, F, and G and that $K \equiv 0$ is the condition for local $(\bar{u},\bar{v})$ with

$$ds^2 = d\bar{u}^2 + d\bar{v}^2.$$

In local coordinates, the ODE for geodesics takes the form

$$\frac{d^2v}{du^2} = a_0(u,v) + 3a_1(u,v)\,\frac{dv}{du} + 3a_2(u,v)\,\left(\frac{dv}{du}\right)^2 + a_3(u,v)\,\left(\frac{dv}{du}\right)^3,$$

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Natural question: Can we recover distances (up to scale) by knowing the shortest curves (i.e., geodesics)? I.e. do a_0, a_1, a_2, and a_3 determine E, F, and G up to a constant multiple?

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Answer: Not always. Central projection from the sphere to the plane takes great circles on the sphere to straight lines in the plane. So knowing which lines are 'straight' doesn't determine distance.

There is also the ‘inverse problem’: When do the solutions of an equation

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Theorem: (2009, B—,Dunajski, Eastwood) There are three conditions $D(a) = 0$ (of order 5) and $E_1(a) = E_2(a) = 0$ (of order 6) that must hold if the above equation describes geodesics of a quadratic form ds^2. Generically, these conditions are sufficient.

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Remark: The proof builds on ideas of S. Lie, R. Liouville, and É. Cartan, but carrying out the proof depended on a combination of modern symbolic manipulation techniques and twistor theory. Most importantly, it depends on being able to interpret the differential equations as geometric objects, so that D, E_1, and E_2 are, in some sense, curvatures of the ‘projective structure’ that the equation defines.

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In local coordinates on some n-dimensional space, a curve $\gamma : [a, b] \to \mathbb{R}^n$ is assigned a length $L(\gamma)$ by an integral

$$L(\gamma) = \int_a^b F\big(\gamma(t), \gamma'(t)\big)\,\mathrm{d}t$$

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Different choices of F define different L-minimizing curves, and hence different notions of ‘straight line’ (geodesics) and ‘distance’ between points.

If the formula

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defines a length that is independent of parametrization, we must have

$$F(x, \lambda \cdot v) = |\lambda|\; F(x, v).$$

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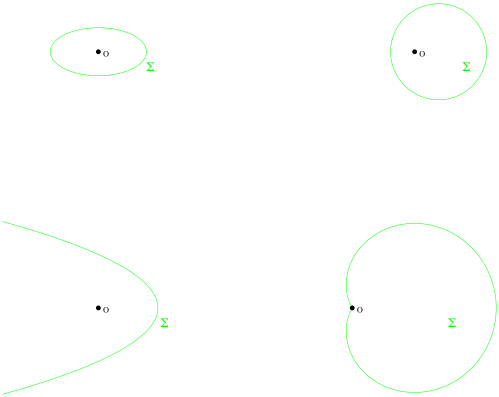
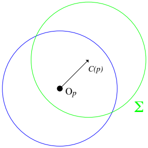
(1) $F(x, v) \geq 0$ is smooth for $v \neq 0$ and
(2) $v \mapsto F(x, v)^2$ is strictly convex for each x.

The convexity condition means that the unit sphere Σ_x at each point should be convex towards the origin:

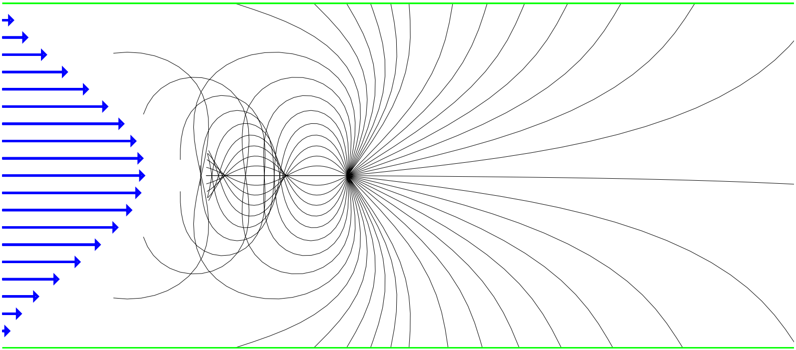
$$\Sigma_x = \{v \mid F(x, v) = 1\}.$$



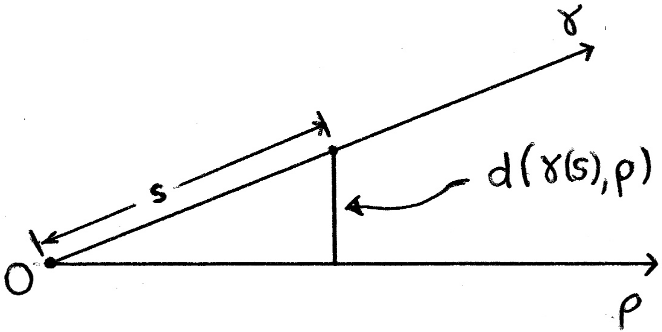
A physical example: River navigation



Some shortest time paths on the river:



Ray Separation: Consider two geodesics rays ρ and γ, emanating from a point O:



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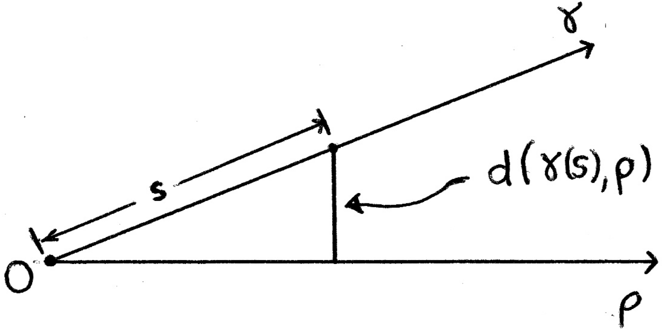
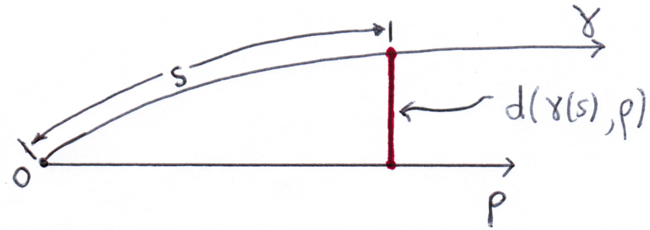


In Euclidean geometry, there is a constant $c(\gamma, \rho) = c(\rho, \gamma)$ so that

$$d\big(\gamma(s), \rho\big) = c(\gamma, \rho)\, s.$$

In the general case, though, the ray separation formula is not so simple:



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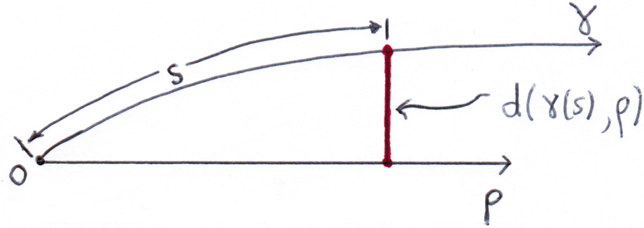

There are functions $c(\gamma,\rho)$ and $K(\gamma,\rho)$ so that

$$d\big(\gamma(s),\rho\big) = c(\gamma,\rho)\,s\,\left(1 - \frac{K(\gamma,\rho)}{6}\,s^2\right) + O\big(s^4, c(\gamma,\rho)^2\big).$$

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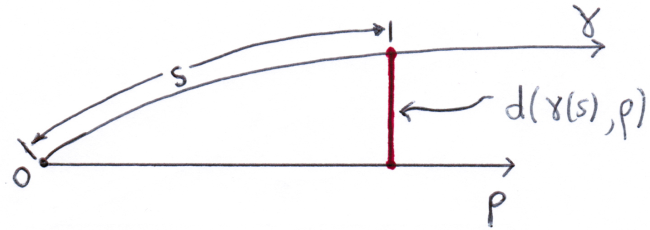

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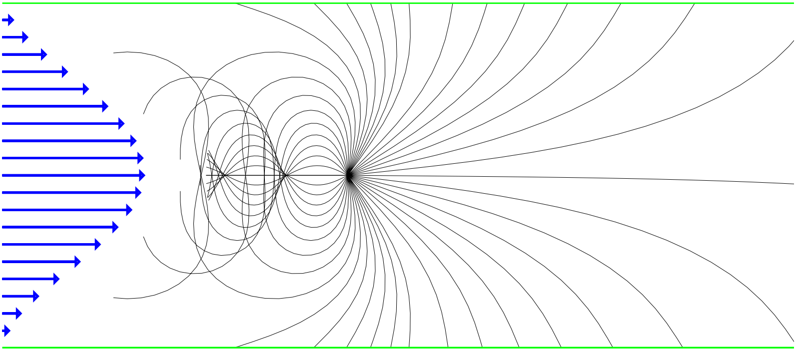
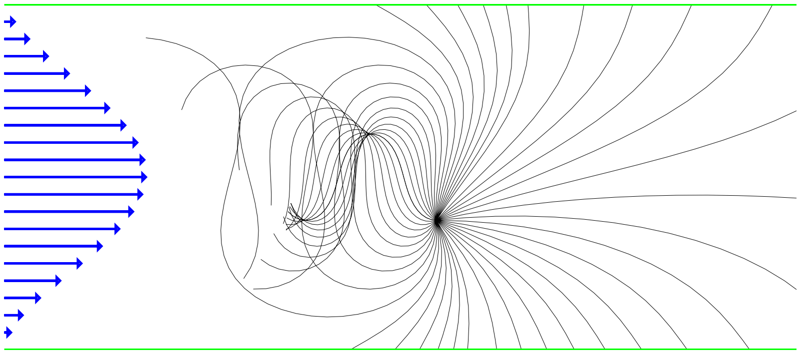
In general, $K(\gamma,\rho) \neq K(\rho,\gamma)$, and $K(\gamma,\rho)$ depends only on the oriented tangent of ρ at O and the plane spanned by the tangents to ρ and γ at O. For this reason, $K(\rho,\gamma)$ is called the flag curvature.

Again, shortest time paths on a river:

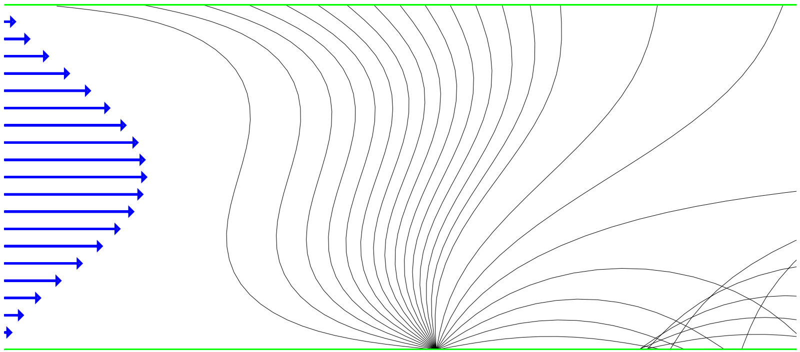


Upstream: $K > 0$ Downstream: $K < 0$

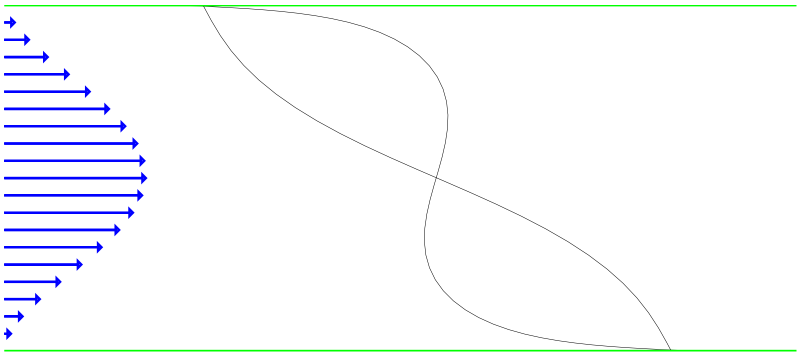
Starting from an off-center point:



Starting on one bank:



The effect of non-reversability:



The shortest path from A to B may not be the shortest path from B to A.

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In this case, Riemann showed that $K(\rho, \gamma)$ depends only on the plane spanned by the tangents to ρ and γ at O. He also showed that, for each constant C, there is a unique Riemannian n-space M_C^n for which $K(\rho, \gamma) = C$ for all geodesic angles. There is always a coordinate chart so that

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Riemannian geometry and its cousin, Lorentzian geometry, have turned out to have many applications in mathematics and physics, from General Relativity to the solution of the Poincaré Conjecture and many more besides.

In the general case, the equation for geodesics in local coordinates $(x, y^1, \ldots, y^n)$ take the form

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This is an active area of research, even today. The problem is how to recognize when a given 'path geometry' can described as the shortest paths according to some metric. While there has been recent progress in describing the differential invariants of a path geometry, *using* those invariants to describe the variational path geometries remains elusive.

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There are now excellent books on the subject, including recent ones by David Bao, S.-S. Chern (who strongly promoted Finsler geometry in the past 20 years), and Zhongmin Shen.

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Classification of the complete Finsler metrics with $K \equiv C$ remains a challenge.

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The Reeb vector field E of ω (i.e., $\omega(E) = 1$ and $d\omega(E, \cdot) = 0$) defines the geodesic flow of Σ; Q^{2n}, its space of integral curves, is the space of geodesics; and $d\omega$ is the pullback to Σ of a symplectic form Ω on Q.

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However, there is a finer structure on Q: If $\ell \in Q$ is a geodesic in M and $x \in \ell$ is a point, then the set $Q_x \subset Q$ of geodesics through x is an Ω-Lagrangian in Q passing through ℓ. The tangent planes $T_\ell(Q_x)$ for $x \in Q$ define an $S^1 \cdot \mathrm{SO}(n)$-substructure B of the $\mathrm{U}(n)$ structure defined by the Kähler structure.

Finally, while the $S^1 \cdot \mathrm{SO}(n)$-substructure B on Q has torsion, it underlies an $S^1 \cdot \mathrm{GL}(n, \mathbb{R})$-structure $\hat{B}$ on Q that is torsion-free.

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Theorem: (B—)

(i) For a generic Finsler structure with $K \equiv 1$, the torsion-free $S^1 \cdot \mathrm{GL}(n, \mathbb{R})$-structure $\hat{B}$ on Q has full holonomy equal to $S^1 \cdot \mathrm{GL}(n, \mathbb{R})$.

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(ii) If $\hat{B}$ on Q is a torsion-free $S^1 \cdot GL(n, \mathbb{R})$-structure on Q whose S^1-curvature form is a positive $(1, 1)$-form, then $\hat{B}$ comes from Finsler structure with $K \equiv 1$ by the above construction.

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Corollary: (B—) There exist torsion-free $S^1 \cdot \mathrm{GL}(n, \mathbb{R})$-structures with full holonomy in dimension n that are not symmetric.

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Remark: This was a holonomy in even dimension that had been previously believed not to exist because it was missed in the holonomy classification project.