Casimir elements and Yangians

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The universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_N)$ is the associative algebra with generators E_{ij} and the defining relations

$$E_{ij} E_{kl} - E_{kl} E_{ij} = \delta_{kj} E_{il} - \delta_{il} E_{kj}.$$

By the Poincaré–Birkhoff–Witt theorem, given any ordering on the set of generators $\{E_{ij}\}$, any element of $\mathrm{U}(\mathfrak{gl}_N)$ can be uniquely written as a linear combination of the ordered monomials in the E_{ij}.

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The center $\mathrm{Z}(\mathfrak{gl}_N)$ of $\mathrm{U}(\mathfrak{gl}_N)$ is

$$\mathrm{Z}(\mathfrak{gl}_N) = \big\{z \in \mathrm{U}(\mathfrak{gl}_N) \mid z\,x = x\,z \quad \text{for all} \quad x \in \mathrm{U}(\mathfrak{gl}_N)\big\}.$$

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The Casimir elements for $\mathfrak{gl}_N$ are elements of $\mathrm{Z}(\mathfrak{gl}_N)$.

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for some complex numbers $\lambda_1, \dots, \lambda_N$.

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This representation is denoted by $L(\lambda)$,

ζ is its highest vector and

$\lambda = (\lambda_1, \dots, \lambda_N)$ is its highest weight.

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Example. $$\chi : E_{11} + \dots + E_{NN} \mapsto \lambda_1 + \dots + \lambda_N$$
$$= l_1 + \dots + l_N - N(N-1)/2.$$

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If u is a complex variable, we set

$$u + E = \begin{bmatrix} u + E_{11} & E_{12} & \dots & E_{1N} \\ E_{21} & u + E_{22} & \dots & E_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ E_{N1} & E_{N2} & \dots & u + E_{NN} \end{bmatrix}.$$

Let $\mathcal{C}(u)$ denote the Capelli determinant

$$\mathcal{C}(u) = \sum_{p \in \mathfrak{S}_N} \operatorname{sgn} p \cdot (u + E)_{p(1),1} \dots (u + E - N + 1)_{p(N),N}.$$

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This is a polynomial in u with coefficients in the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_N)$,

$$\mathcal{C}(u) = u^N + \mathcal{C}_1 \, u^{N-1} + \dots + \mathcal{C}_N, \qquad \mathcal{C}_i \in \mathrm{U}(\mathfrak{gl}_N).$$

Example. For $N = 2$ we have

$$\begin{aligned}\mathcal{C}(u) &= (u + E_{11})\,(u + E_{22} - 1) - E_{21}\,E_{12} \\ &= u^2 + (E_{11} + E_{22} - 1)\,u + E_{11}\,(E_{22} - 1) - E_{21}\,E_{12}.\end{aligned}$$

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Note that

$$\mathcal{C}_1 = E_{11} + E_{22} - 1, \qquad \mathcal{C}_2 = E_{11}\,(E_{22} - 1) - E_{21}\,E_{12}$$

are Casimir elements for $\mathfrak{gl}_2$ and

$$\chi(\mathcal{C}_1) = l_1 + l_2,$$
$$\chi(\mathcal{C}_2) = l_1\,l_2.$$

Theorem (C, HU).

The coefficients $\mathcal{C}_1, \dots, \mathcal{C}_N$ belong to $\mathrm{Z}(\mathfrak{gl}_N)$.

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Hence, $\chi(\mathcal{C}_k)$ is the elementary symmetric polynomial of degree k in $l_1, \dots, l_N$,

$$\chi(\mathcal{C}_k) = \sum_{i_1 < \dots < i_k} l_{i_1} \dots l_{i_k}.$$

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$$\chi(\mathcal{C}_k) = \sum_{i_1 < \dots < i_k} l_{i_1} \dots l_{i_k}.$$

Moreover, $\mathrm{Z}(\mathfrak{gl}_N)$ is the algebra of polynomials in $\mathcal{C}_1, \dots, \mathcal{C}_N$.

Hudson elements

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Given any complex numbers $a_1, \ldots, a_N$, set

$$H(a_1, \ldots, a_N) = \frac{1}{N!} \sum_{p,q \in \mathfrak{S}_N} \operatorname{sgn} p \cdot \operatorname{sgn} q \cdot (a_1 + E)_{p(1),q(1)} \ldots (a_N + E)_{p(N),q(N)}.$$

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Theorem (H, IU).

The Capelli determinant can be written as

$$\mathcal{C}(u) = H(u, u-1, \ldots, u-N+1).$$

Gelfand invariants

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These are the elements of $\mathrm{U}(\mathfrak{gl}_N)$ defined by

$$\operatorname{tr} E^k = \sum_{i_1, i_2, \dots, i_k = 1}^{N} E_{i_1 i_2} E_{i_2 i_3} \dots E_{i_k i_1}, \qquad k = 0, 1, \dots .$$

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Example. For $N = 2$ we have

$$\mathrm{tr}\, E = E_{11} + E_{22},$$

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Note that they are Casimir elements and

$$\chi(\operatorname{tr} E) = l_1 + l_2 - 1,$$

$$\chi(\operatorname{tr} E^2) = l_1^2 + l_2^2 + l_1 + l_2.$$

Theorem (Newton's formula). We have

$$1+\sum_{k=0}^{\infty}\frac{(-1)^k\,\mathrm{tr}\,E^k}{(u-N+1)^{k+1}}=\frac{\mathcal{C}(u+1)}{\mathcal{C}(u)}.$$

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Proof.

This is equivalent to the Perelomov–Popov formulas

$$1+\sum_{k=0}^{\infty}\frac{(-1)^k\,\chi(\operatorname{tr}E^k)}{(u-N+1)^{k+1}}=\prod_{i=1}^{N}\frac{u+l_i+1}{u+l_i}.$$

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The following identities hold for the image of the matrix E in the representation $L(\lambda)$ of $\mathfrak{gl}_N$:

$$\prod_{i=1}^{N}(E-l_i-N+1)=0 \qquad \text{and} \qquad \prod_{i=1}^{N}(E^t-l_i)=0.$$

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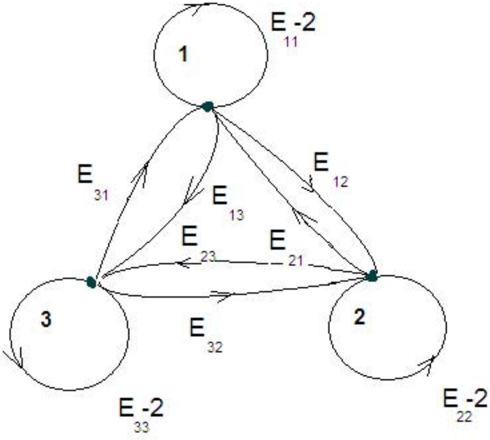
Given a path in the graph, take the ordered product of the labels of the arrows to get an element of $\mathrm{U}(\mathfrak{gl}_N)$ which we call the label of the path.

Example. The complete oriented graph for $m = 3$:



For any positive integer k set

$$\Phi_k^{(m)} = \sum \frac{k}{\sharp \text{ returns to } m} \Big\{ \text{label of the path} \Big\},$$

summed over all paths in the graph from m to m of length k.

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Example.

$$\Phi_1^{(m)} = E_{mm} - m + 1$$

$$\Phi_2^{(m)} = (E_{mm} - m + 1)^2 + 2 \sum_{i=1}^{m-1} E_{m\,i} E_{i\,m}.$$

Theorem (GKLLRT). For any $k \geqslant 1$ the element

$$\Phi_k = \Phi_k^{(1)} + \cdots + \Phi_k^{(N)}$$

belongs to $\mathrm{Z}(\mathfrak{gl}_N)$. Moreover,

$$\chi(\Phi_k) = l_1^k + \cdots + l_N^k.$$

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For $N = 2n$ or $N = 2n + 1$, respectively, set

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We will number the rows and columns of $N \times N$ matrices by the indices $\{-n, \ldots, -1, 0, 1, \ldots, n\}$ if $N = 2n + 1$, and by $\{-n, \ldots, -1, 1, \ldots, n\}$ if $N = 2n$.

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The Lie algebra $\mathfrak{g}_N = \mathfrak{o}_N$ is spanned by the elements

$$F_{ij} = E_{ij} - E_{-j,-i}, \qquad -n \leqslant i, j \leqslant n.$$

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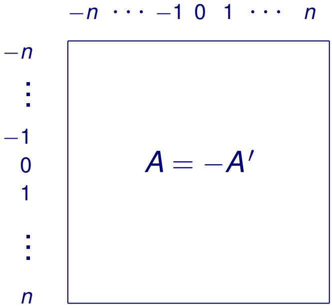

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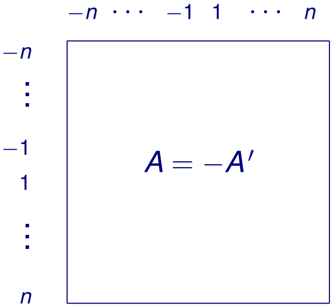

Skew-symmetric matrices with respect to the second diagonal.

The Lie algebra $\mathfrak{g}_N = \mathfrak{sp}_N$ with $N = 2n$ is spanned by the elements

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The Lie algebra $\mathfrak{g}_N = \mathfrak{sp}_N$ with $N = 2n$ is spanned by the elements

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$$\begin{array}{c|c|c|} & -n \ \cdots \ -1 & 1 \ \cdots \ n \\ \hline -n,\ \dots,\ -1 & A & B = B' \\ \hline 1,\ \dots,\ n & C = C' & -A' \\ \hline \end{array}$$

For any n-tuple of complex numbers $\lambda = (\lambda_1, \dots, \lambda_n)$ the corresponding irreducible highest weight representation $V(\lambda)$ of $\mathfrak{g}_N$ is generated by a nonzero vector ξ such that

$$F_{ij}\,\xi = 0 \qquad \text{for} \quad -n \leqslant i < j \leqslant n, \qquad \text{and}$$

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$$\rho_i = -\rho_{-i} = \begin{cases} -i+1 & \text{for} \quad \mathfrak{g}_N = \mathfrak{o}_{2n}, \\ -i+\frac{1}{2} & \text{for} \quad \mathfrak{g}_N = \mathfrak{o}_{2n+1}, \\ -i & \text{for} \quad \mathfrak{g}_N = \mathfrak{sp}_{2n}, \end{cases}$$

for $i = 1, \dots, n$. Also, $\rho_0 = 1/2$ in the case $\mathfrak{g}_N = \mathfrak{o}_{2n+1}$.

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In the D case $\chi(z)$ is the sum of a symmetric polynomial in $l_1^2, \dots, l_n^2$ and $l_1 \dots l_n$ times a symmetric polynomial in $l_1^2, \dots, l_n^2$.

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Example. For $\mathfrak{g}_N = \mathfrak{o}_N$

$$\sum_{m=1}^{n} \Big((F_{mm} + \rho_m)^2 + 2 \sum_{-m<i<m} F_{mi} F_{im} \Big)$$

is the second degree Casimir element. Its Harish-Chandra image is

$$l_1^2 + \cdots + l_n^2 .$$

Capelli-type determinant for $\mathfrak{g}_N$

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Introduce a special map

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from the symmetric group $\mathfrak{S}_N$ into itself.

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Given a set of positive integers $a_1 < \cdots < a_N$ we regard $\mathfrak{S}_N$ as the group of their permutations.

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$$\Big\{(a_k, a_l) \mid k \neq l\Big\}$$

into itself by the rule

$$\begin{aligned}
(a_k, a_l) &\mapsto (a_l, a_k), && k, l < N,\\
(a_k, a_N) &\mapsto (a_{N-1}, a_k), && k < N-1,\\
(a_N, a_k) &\mapsto (a_k, a_{N-1}), && k < N-1,
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\end{aligned}$$

Let $p = (p_1, \ldots, p_N)$ be a permutation of the indices $a_1, \ldots, a_N$.

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Then the pair (p'_2, p'_{N-2}) is found as the image of (p_2, p_{N-1}) under the above map, etc.

Example.

$p = (3, 5, 7, 6, 1, 2, 4)$.

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Thus, $\quad p' = (4, 2, 1, 6, 5, 3, 7)$.

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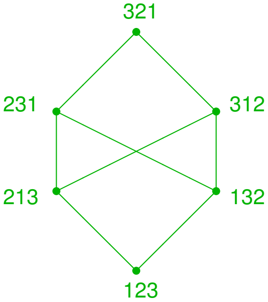

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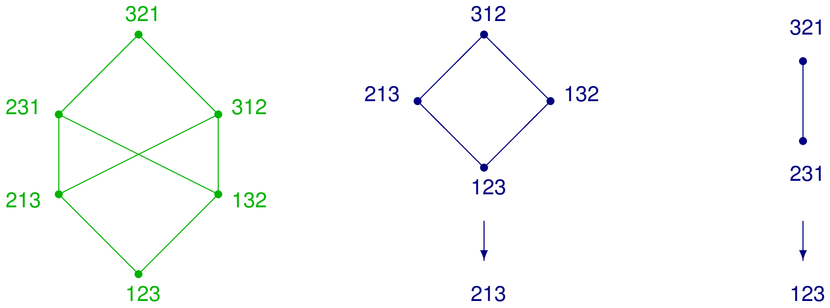

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If u is a complex variable, we set

$$u + F = \begin{bmatrix} u + F_{-n,-n} & F_{-n,-n+1} & \cdots & F_{-n,n} \\ F_{-n+1,-n} & u + F_{-n+1,-n+1} & \cdots & F_{-n+1,n} \\ \vdots & \vdots & \ddots & \vdots \\ F_{n,-n} & F_{n,-n+1} & \cdots & u + F_{n,n} \end{bmatrix}.$$

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Note that

$$F_{-j,-i} = \begin{cases} -F_{ij} & \text{in the orthogonal case,} \\ -\operatorname{sgn} i \cdot \operatorname{sgn} j \cdot F_{ij} & \text{in the symplectic case.} \end{cases}$$

Introduce the Capelli-type determinant

$$\mathcal{C}(u) = (-1)^n \sum_{p \in \mathfrak{S}_N} \operatorname{sgn} pp' \cdot (u + \rho_{-n} + F)_{-b_{p(1)}, b_{p'(1)}} \times \cdots \times (u + \rho_n + F)_{-b_{p(N)}, b_{p'(N)}},$$

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where $(b_1, \dots, b_N)$ is a fixed permutation of the indices $(-n, \dots, n)$ and p' is the image of p under the map φ_N.

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$$\chi : \mathcal{C}(u) \mapsto \prod_{i=1}^{n} (u^2 - l_i^2), \qquad \text{if} \quad N = 2n,$$

and

$$\chi : \mathcal{C}(u) \mapsto \Big(u + \frac{1}{2}\Big) \prod_{i=1}^{n} (u^2 - l_i^2), \qquad \text{if} \quad N = 2n + 1.$$

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$$\begin{aligned}\mathcal{C}(u) = {} & (u + F_{-1,-1} + 1/2)\,(u + 1/2)\,(u + F_{11} - 1/2) \\ & - F_{0,-1}\,F_{-1,0}\,(u + F_{11} - 1/2) \\ & - F_{10}\,(u + F_{-1,-1} + 1/2)\,F_{01}.\end{aligned}$$

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Noncommutative Pfaffians and Hafnians

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The Pfaffian $\operatorname{Pf} A$ of a $2k \times 2k$ matrix $A = [A_{ij}]$ is defined by

$$\operatorname{Pf} A = \frac{1}{2^k k!} \sum_{\sigma \in \mathfrak{S}_{2k}} \operatorname{sgn} \sigma \cdot A_{\sigma(1),\sigma(2)} \dots A_{\sigma(2k-1),\sigma(2k)}.$$

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If A is a skew-symmetric numerical matrix, then

$$\det A = (\mathrm{Pf}\, A)^2.$$

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$$\mathrm{Pf}\begin{bmatrix} 0 & A_{12} & A_{13} & A_{14} \\ -A_{12} & 0 & A_{23} & A_{24} \\ -A_{13} & -A_{23} & 0 & A_{34} \\ -A_{14} & -A_{24} & -A_{34} & 0 \end{bmatrix} = A_{12}A_{34} - A_{13}A_{24} + A_{14}A_{23}.$$

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Let $\mathfrak{g}_N = \mathfrak{o}_N$. For any subset I of $\{-n, \ldots, n\}$ containing $2k$ elements $i_1 < \cdots < i_{2k}$, the submatrix

$$F_I = \begin{bmatrix} 0 & F_{i_1,-i_2} & \ldots & F_{i_1,-i_{2k}} \\ F_{i_2,-i_1} & 0 & \ldots & F_{i_2,-i_{2k}} \\ \vdots & \vdots & \ddots & \vdots \\ F_{i_{2k},-i_1} & F_{i_{2k},-i_2} & \ldots & 0 \end{bmatrix}$$

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Set

$$C_k = (-1)^k \cdot \sum_I \mathrm{Pf}\, F_I \cdot \mathrm{Pf}\, F_{I^*}, \qquad I^* = \{-i_{2k}, \dots, -i_1\},$$

summed over all subsets I with $|I| = 2k$.

Theorem (MN). $C_1, \dots, C_n$ are Casimir elements for $\mathfrak{o}_N$.

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Corollary.

$$\frac{\mathcal{C}(u)}{(u + \rho_{-n}) \dots (u + \rho_n)} = 1 + \sum_{k=1}^{n} \frac{C_k}{(u^2 - \rho_{n-k+1}^2) \dots (u^2 - \rho_n^2)}.$$

For any $k \geqslant 1$ let $I = \{i_1, \ldots, i_{2k}\}$ be a multiset whose elements belong to $\{-n, \ldots, n\}$.

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The Hafnian $\mathrm{Hf}\, A_I$ of the matrix A_I is defined by

$$\mathrm{Hf}\, A_I = \frac{1}{2^k k!} \sum_{\sigma \in \mathfrak{S}_{2k}} A_{i_{\sigma(1)}, i_{\sigma(2)}} \dots A_{i_{\sigma(2k-1)}, i_{\sigma(2k)}}.$$

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Remark. The term is due to Caianiello, '56. Hafnia is the Latin name for "Copenhagen"; cf. Hafnium[72].

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is symmetric. Set

$$D_k = \sum_I \frac{\operatorname{sgn}(i_1 \dots i_{2k})}{\alpha_{-n}! \dots \alpha_n!} \cdot \operatorname{Hf} F_I \cdot \operatorname{Hf} F_{I^*}, \qquad I^* = \{-i_{2k}, \dots, -i_1\},$$

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Corollary.

$$\Big(\frac{\mathcal{C}(u)}{(u + \rho_{-n}) \ldots (u + \rho_n)} \Big)^{-1} = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k D_k}{(u^2 - (n+1)^2) \ldots (u^2 - (n+k)^2)}.$$

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More generally, we have

$$\big[(E^{r+1})_{ij}, (E^s)_{kl}\big] - \big[(E^r)_{ij}, (E^{s+1})_{kl}\big] = (E^r)_{kj}(E^s)_{il} - (E^s)_{kj}(E^r)_{il},$$

where $r, s \geqslant 0$ and $E^0 = 1$ is the identity matrix.

Yangian for $\mathfrak{gl}_N$

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Definition

The Yangian for $\mathfrak{gl}_N$ is the associative algebra over $\mathbb{C}$ with countably many generators $t_{ij}^{(1)},\ t_{ij}^{(2)},\dots$ where $i,j=1,\dots,N$, and the defining relations

$$[t_{ij}^{(r+1)},t_{kl}^{(s)}]-[t_{ij}^{(r)},t_{kl}^{(s+1)}]=t_{kj}^{(r)}t_{il}^{(s)}-t_{kj}^{(s)}t_{il}^{(r)},$$

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This algebra is denoted by $\mathrm{Y}(\mathfrak{gl}_N)$.

Introduce the formal generating series

$$t_{ij}(u) = \delta_{ij} + t_{ij}^{(1)} u^{-1} + t_{ij}^{(2)} u^{-2} + \cdots \in \mathrm{Y}(\mathfrak{gl}_N)[[u^{-1}]].$$

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The defining relations are also equivalent to

$$[t_{ij}^{(r)}, t_{kl}^{(s)}] = \sum_{a=1}^{\min\{r,s\}} \Big(t_{kj}^{(a-1)} t_{il}^{(r+s-a)} - t_{kj}^{(r+s-a)} t_{il}^{(a-1)} \Big).$$

Evaluation homomorphism

Proposition. The assignment

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defines a surjective homomorphism $\mathrm{Y}(\mathfrak{gl}_N) \to \mathrm{U}(\mathfrak{gl}_N)$.

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Hence, we may regard $\mathrm{U}(\mathfrak{gl}_N)$ as a subalgebra of $\mathrm{Y}(\mathfrak{gl}_N)$.

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Introduce the $N \times N$ matrix $T(u)$ whose ij-th entry is the series $t_{ij}(u)$. We regard $T(u)$ as an element of the algebra $\mathrm{End}\,\mathbb{C}^N \otimes \mathrm{Y}(\mathfrak{gl}_N)[[u^{-1}]]$:

$$T(u) = \sum_{i,j=1}^{N} e_{ij} \otimes t_{ij}(u),$$

where $e_{ij} \in \mathrm{End}\,\mathbb{C}^N$ are the standard matrix units.

For any positive integer m consider the algebra

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For any $a \in \{1, \dots, m\}$ denote by $T_a(u)$ the matrix $T(u)$ which corresponds to the a-th copy of the algebra $\mathrm{End}\,\mathbb{C}^N$ in the tensor product algebra.

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For any $a \in \{1, \dots, m\}$ denote by $T_a(u)$ the matrix $T(u)$ which corresponds to the a-th copy of the algebra $\mathrm{End}\,\mathbb{C}^N$ in the tensor product algebra. That is, $T_a(u)$ is a formal power series in u^{-1} given by

$$T_a(u) = \sum_{i,j=1}^{N} \underbrace{1 \otimes \cdots \otimes 1}_{a-1} \otimes e_{ij} \otimes \underbrace{1 \otimes \cdots \otimes 1}_{m-a} \otimes t_{ij}(u),$$

where 1 is the identity matrix.

Similarly, if

$$C = \sum_{i,j,k,l=1}^{N} c_{ijkl}\, e_{ij} \otimes e_{kl} \in \mathrm{End}\,\mathbb{C}^N \otimes \mathrm{End}\,\mathbb{C}^N,$$

Similarly, if

$$C = \sum_{i,j,k,l=1}^{N} c_{ijkl}\, e_{ij} \otimes e_{kl} \in \mathrm{End}\,\mathbb{C}^N \otimes \mathrm{End}\,\mathbb{C}^N,$$

then for any two indices $a, b \in \{1, \dots, m\}$ such that $a < b$, define the element C_{ab} of the algebra $(\mathrm{End}\,\mathbb{C}^N)^{\otimes m}$ by

$$C_{ab} = \sum_{i,j,k,l=1}^{N} c_{ijkl}\, \underbrace{1 \otimes \cdots \otimes 1}_{a-1} \otimes e_{ij} \otimes \underbrace{1 \otimes \cdots \otimes 1}_{b-a-1} \otimes e_{kl} \otimes \underbrace{1 \otimes \cdots \otimes 1}_{m-b}.$$

Consider now the permutation operator

$$P = \sum_{i,j=1}^{N} e_{ij} \otimes e_{ji} \in \operatorname{End} \mathbb{C}^N \otimes \operatorname{End} \mathbb{C}^N.$$

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The rational function

$$R(u) = 1 - P u^{-1}$$

with values in $\mathrm{End}\,\mathbb{C}^N \otimes \mathrm{End}\,\mathbb{C}^N$ is called the Yang R-matrix.

Proposition. We have the identity

$$R_{12}(u)\, R_{13}(u+v)\, R_{23}(v) = R_{23}(v)\, R_{13}(u+v)\, R_{12}(u).$$

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This relation is known as the Yang–Baxter equation. The Yang R-matrix is its simplest nontrivial solution.

Proposition. The defining relations of the algebra $\mathrm{Y}(\mathfrak{gl}_N)$ can be written in the equivalent form

$$R(u-v)\, T_1(u)\, T_2(v) = T_2(v)\, T_1(u)\, R(u-v).$$

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The matrix relation is called the *RTT* relation (or ternary relation).

Quantum determinant

Quantum determinant

For any $m \geqslant 2$ introduce the rational function $R(u_1, \dots, u_m)$ with values in the tensor product algebra $(\mathrm{End}\,\mathbb{C}^N)^{\otimes m}$ by

$$R(u_1, \dots, u_m) = (R_{m-1,m})(R_{m-2,m}R_{m-2,m-1}) \dots (R_{1m} \dots R_{12}),$$

where $u_1, \dots, u_m$ are independent complex variables and

$$R_{ij} = R_{ij}(u_i - u_j) = 1 - P_{ij}(u_i - u_j)^{-1}.$$

.

Applying the *RTT* relation repeatedly,

we come to the fundamental relation

$$R(u_1,\dots,u_m)\,T_1(u_1)\dots T_m(u_m)=T_m(u_m)\dots T_1(u_1)\,R(u_1,\dots,u_m).$$

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Lemma (Jucys). If $u_i - u_{i+1} = 1$ for all $i = 1,\dots,m-1$ then

$$R(u_1,\dots,u_m) = A_m,$$

the image of the anti-symmetrizer $\sum_{p\in\mathfrak{S}_m} \operatorname{sgn} p\cdot p \in \mathbb{C}[\mathfrak{S}_m]$ in the algebra $\operatorname{End}(\mathbb{C}^N)^{\otimes m}$.

Hence, taking $m = N$ we get

$$A_N \, T_1(u) \dots T_N(u - N + 1) = T_N(u - N + 1) \dots T_1(u) \, A_N.$$

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Definition

The quantum determinant of the matrix $T(u)$ with the coefficients in $\mathrm{Y}(\mathfrak{gl}_N)$ is the formal series

$$\mathrm{qdet}\, T(u) = 1 + d_1 u^{-1} + d_2 u^{-2} + \dots$$

such that both sides of the above relation are equal to $A_N\, \mathrm{qdet}\, T(u)$.

We have

$$\operatorname{qdet} T(u) = \sum_{p \in \mathfrak{S}_N} \operatorname{sgn} p \cdot t_{p(1),1}(u) \dots t_{p(N),N}(u - N + 1)$$

$$= \sum_{p \in \mathfrak{S}_N} \operatorname{sgn} p \cdot t_{1,p(1)}(u - N + 1) \dots t_{N,p(N)}(u).$$

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Example. For $N=2$ we have

$$\operatorname{qdet} T(u) = t_{11}(u)\, t_{22}(u-1) - t_{21}(u)\, t_{12}(u-1)$$

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Center of $\mathrm{Y}(\mathfrak{gl}_N)$

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Theorem (KS). The coefficients $d_1, d_2, \dots$ of the series $\mathrm{qdet}\, T(u)$ belong to the center $\mathrm{ZY}(\mathfrak{gl}_N)$ of the algebra $\mathrm{Y}(\mathfrak{gl}_N)$.

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Note that

$$\mathcal{C}(u) = u(u-1)\dots(u-N+1)\,\pi_N\big(\mathrm{qdet}\, T(u)\big).$$

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Note that

$$\mathcal{C}(u) = u(u-1)\dots(u-N+1)\,\pi_N\big(\mathrm{qdet}\, T(u)\big).$$

Corollary. All coefficients of $\mathcal{C}(u)$ are Casimir elements for $\mathfrak{gl}_N$.

Quantum Liouville formula

Quantum Liouville formula

Consider the series $z(u)$ with coefficients from $\mathrm{Y}(\mathfrak{gl}_N)$ given by the formula

$$z(u)^{-1} = \frac{1}{N}\,\mathrm{tr}\,\Big(T(u)\,T^{-1}(u-N)\Big),$$

so that

$$z(u) = 1 + z_2 u^{-2} + z_3 u^{-3} + \dots \quad \text{where} \quad z_i \in \mathrm{Y}(\mathfrak{gl}_N).$$

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Theorem (N). We have the relation

$$z(u) = \frac{\mathrm{qdet}\, T(u-1)}{\mathrm{qdet}\, T(u)}.$$

Application to $\mathfrak{gl}_N$

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Recall the evaluation homomorphism $\pi_N : T(u) \mapsto 1 + E\, u^{-1}$:

$$\pi_N : z(-u+N)^{-1} \mapsto \frac{1}{N}\,\mathrm{tr}\Big((1 - E\,(u-N)^{-1})(1 - E\,u^{-1})^{-1}\Big)$$
$$= 1 - \frac{1}{u-N} \sum_{k=1}^{\infty} \mathrm{tr}\, E^k\, u^{-k}.$$

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The quantum Liouville formula gives

$$z(u+1)^{-1} = \frac{\mathrm{qdet}\,T(u+1)}{\mathrm{qdet}\,T(u)}.$$

Applying the evaluation homomorphism to both sides of this relation, we get Newton's formula.

Twisted Yangians

Twisted Yangians

Consider the orthogonal Lie algebra $\mathfrak{o}_N$ as the subalgebra of $\mathfrak{gl}_N$ spanned by the skew-symmetric matrices. The elements $F_{ij} = E_{ij} - E_{ji}$ with $i < j$ form a basis of $\mathfrak{o}_N$. Introduce the $N \times N$ matrix F whose ij-th entry is F_{ij}.

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The matrix elements of the powers of the matrix F are known to satisfy the relations

$$[F_{ij}, (F^s)_{kl}] = \delta_{kj}(F^s)_{il} - \delta_{il}(F^s)_{kj} - \delta_{ik}(F^s)_{jl} + \delta_{lj}(F^s)_{ki}.$$

Introduce the generating series

$$f_{ij}(u) = \delta_{ij} + \sum_{r=1}^{\infty} (F^r)_{ij} \Big(u + \frac{N-1}{2} \Big)^{-r}.$$

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Then we have the relations

$$\begin{aligned}(u^2 - v^2)\,[f_{ij}(u), f_{kl}(v)] &= (u+v)\,\big(f_{kj}(u)\,f_{il}(v) - f_{kj}(v)\,f_{il}(u)\big) \\ &- (u-v)\,\big(f_{ik}(u)\,f_{jl}(v) - f_{ki}(v)\,f_{lj}(u)\big) \\ &+ f_{ki}(u)\,f_{jl}(v) - f_{ki}(v)\,f_{jl}(u).\end{aligned}$$

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Let $G = [g_{ij}]$ be a nonsingular (skew-)symmetric matrix.

The twisted Yangian $\mathrm{Y}(\mathfrak{g}_N)$ is an associative algebra with generators $s_{ij}^{(1)},\ s_{ij}^{(2)},\dots$ where $1 \leqslant i, j \leqslant N$, and the defining relations written in terms of the generating series

$$s_{ij}(u) = g_{ij} + s_{ij}^{(1)} u^{-1} + s_{ij}^{(2)} u^{-2} + \dots$$

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and

$$s_{ji}(-u)=\pm\,s_{ij}(u)+\frac{s_{ij}(u)-s_{ij}(-u)}{2u}.$$

Matrix form of the defining relations

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Introduce the $N \times N$ matrix $S(u)$ by

$$S(u) = \sum_{i,j=1}^{N} e_{ij} \otimes s_{ij}(u) \in \mathrm{End}\, \mathbb{C}^N \otimes \mathrm{Y}(\mathfrak{g}_N)[[u^{-1}]]$$

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The defining relations of $\mathrm{Y}(\mathfrak{g}_N)$ have the form

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and

$$S^t(-u) = \pm\, S(u) + \frac{S(u) - S(-u)}{2u}.$$

Here

$$R(u) = 1 - P u^{-1}$$

is the Yang R-matrix, while

$$R^t(u) = 1 - Q\, u^{-1}, \qquad Q = \sum_{i,j=1}^{N} e_{ij} \otimes e_{ij}.$$

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The mapping

$$S(u) \mapsto T(u)\, G\, T^{\, t}(-u)$$

defines an embedding $\mathrm{Y}(\mathfrak{g}_N) \hookrightarrow \mathrm{Y}(\mathfrak{gl}_N)$.

Sklyanin determinant

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The Sklyanin determinant is a series in u^{-1} defined by

$$\operatorname{sdet} S(u) = \gamma_{n,G}(u) \operatorname{qdet} T(u) \operatorname{qdet} T(-u+N-1),$$

where

$$\gamma_{n,G}(u) = \begin{cases} \det G & \text{if} \quad \mathfrak{g}_N = \mathfrak{o}_N, \\ \dfrac{2u+1}{2u-2n+1} \det G & \text{if} \quad \mathfrak{g}_N = \mathfrak{sp}_{2n}. \end{cases}$$

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All coefficients of $\operatorname{sdet} S(u)$ are contained in $\mathrm{Y}(\mathfrak{g}_N)$ and belong to the center of $\mathrm{Y}(\mathfrak{g}_N)$.

Introduce the scalar $\gamma_n(u)$ by

$$\gamma_n(u) = \begin{cases} 1 & \text{if} \quad \mathfrak{g}_N = \mathfrak{o}_N, \\ (-1)^n \dfrac{2u+1}{2u-2n+1} & \text{if} \quad \mathfrak{g}_N = \mathfrak{sp}_{2n}. \end{cases}$$

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Theorem (M). We have

$$\begin{aligned} &\operatorname{sdet} S(u) \\ &\qquad = \gamma_n(u) \sum_{p \in \mathfrak{S}_N} \operatorname{sgn} pp' \cdot s^t_{p(1),p'(1)}(-u) \dots s^t_{p(n),p'(n)}(-u+n-1) \\ &\qquad\qquad \times s_{p(n+1),p'(n+1)}(u-n) \dots s_{p(N),p'(N)}(u-N+1). \end{aligned}$$

Examples. For $N = 2$ we have

$$\operatorname{sdet} S(u) = \frac{1 \mp 2u}{1 - 2u} \big(s^t_{11}(-u)\, s_{22}(u-1) - s^t_{21}(-u)\, s_{12}(u-1) \big).$$

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If $\quad N = 3 \quad$ then $\quad \operatorname{sdet} S(u) =$

$$s^t_{22}(-u)\, s_{11}(u-1)\, s_{33}(u-2) + s^t_{12}(-u)\, s_{31}(u-1)\, s_{23}(u-2)$$
$$+ s^t_{21}(-u)\, s_{32}(u-1)\, s_{13}(u-2) - s^t_{12}(-u)\, s_{21}(u-1)\, s_{33}(u-2)$$
$$- s^t_{32}(-u)\, s_{11}(u-1)\, s_{23}(u-2) - s^t_{31}(-u)\, s_{22}(u-1)\, s_{13}(u-2).$$