Sydney University Mathematical Society Problem Competition 2013

This competition is open to undergraduates (including Honours students) at any Australian university or tertiary institution. Those who enter do so as individuals, and must **not** receive help with the problems, e.g. from fellow students, lecturers or online groups.

Entrants may submit solutions to as many problems as they wish. Prizes ($60 book vouchers from the Co-op Bookshop) will be awarded for the best solution to each of the 10 problems. Students from the University of Sydney are also eligible for the Norbert Quirk Prizes, based on the overall quality of their entry (one for each of 1st, 2nd and 3rd years).

Extensions and generalizations of any problem are invited and are taken into account when assessing solutions. If two or more solutions to a problem are essentially equal, the prize will be given to the student(s) in the earlier year of university. If a problem receives no complete solutions, partial solutions may suffice.

Entries must be received by **Wednesday, August 14, 2013**. They may be posted to Associate Professor Anthony Henderson, School of Mathematics and Statistics, The University of Sydney, NSW 2006, or handed in to Carslaw room 805. Please mark your entry SUMS Problem Competition 2013, and include your name, university, student number, year of study, and postal address (or email address for University of Sydney students) for the awarding of prizes. Please note that entries will not be returned.

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1. Alice, Bess and Cath want to host seven parties in 2014, occurring on the seven different days of the week. Alice writes down a list of seven dates in 2014, in which no day of the week is repeated. Bess crosses out the first date on Alice's list (a Wednesday) and replaces it with a randomly chosen alternative date, without looking at the rest of the list. Cath now considers each of the other dates on the list in turn, starting from the second one. If its day of the week has not appeared earlier in the list, she leaves the date unchanged; and if its day of the week has appeared earlier, she replaces it with a randomly chosen date whose day of the week has not appeared earlier. What is the probability that she changes the last date on the list?

2. David is also planning a party, at which there are to be 26 guests. Considering that a triple of guests has 'social potential' if it contains a pair who have met each other before and also a pair who haven't met each other before, he wants at least half of all the $\binom{26}{3}$ triples to have social potential. What is the smallest number of previously-acquainted pairs that could possibly be compatible with this requirement?

3. Define a sequence by the initial value $b_1 = 3$ and the recurrence relation $b_{n+1} = b_n^2 - 2$ for $n \geq 1$. Evaluate the limit $\displaystyle\lim_{m\to\infty} \frac{b_m}{b_1 b_2 \cdots b_{m-1}}$.

4. Find the determinant of the $n \times n$ matrix whose (i, j)-entry is i if $i \neq j$, and is $i + 1$ if $i = j$.

5. Fix an integer $n > 1$ and consider the permutations of the set $\{1, 2, \cdots, n\}$. Say that such a permutation σ is *self-inverse* if $\sigma(\sigma(i)) = i$ for all $1 \leq i \leq n$. Say that σ is *modest* if

$\sigma(i) > \min\{\sigma(i+1), \sigma(i+2)\}$ for all $1 \le i \le n-2$. Prove that the number of self-inverse permutations equals the number of modest permutations.

6. Let $f(x)$ be a polynomial function with rational coefficients and degree at least 2. Show that there are infinitely many rational numbers that are not equal to $f(x)$ for any rational x.

7. For any positive integer n, let A_n be the 2×2 matrix $\begin{bmatrix} \frac{n}{n+1} & 2 \\ \frac{2n}{n+1} & 1 \end{bmatrix}$. Define $B_n = A_1 A_2 \cdots A_n$. Show that the entries in the bottom row of B_n are integers.

8. Let a and c be positive real numbers. Evaluate

$$\int_{c-i\infty}^{c+i\infty} \frac{a^z}{z^2}\, dz.$$

Here $\int_{c-i\infty}^{c+i\infty}$ denotes a contour integral in the complex plane, along the vertical line $\operatorname{Re}(z) = c$ traversed upwards.

9. Let $\mathbb{N} = \{0, 1, 2, \cdots\}$. Call a subset $A \subseteq \mathbb{N}$ *special* if it satisfies:

a) $0 \in A$,

b) $a \in A \Rightarrow a + 10 \in A$,

c) $a \in A \Rightarrow a + 2013 \in A$.

How many special subsets $A \subseteq \mathbb{N}$ are there?

10. Let A, B, C, D be matrices over the complex numbers satisfying the following conditions:

a) A, B are $n \times n$ matrices that are *nilpotent* (so $A^n = B^n$ is the zero matrix);

b) C is an $n \times 2$ matrix and D is a $2 \times n$ matrix;

c) $A + B = CD$.

Show that $DA^iC + (-1)^{i+1} DB^iC$ is a scalar matrix for all nonnegative integers i.