Sydney University Mathematical Society
Problems Competition 2005

This competition is open to all undergraduates at any Australian university or tertiary institution. Prizes ($50 book vouchers from the Co-op Bookshop) will be awarded for the best correct solution to each of the 10 problems. Entrants from the University of Sydney will also be eligible for the Norbert Quirk Prizes (one for each of 1st, 2nd and 3rd years). Entries from fourth year students will be considered. When prizewinners are being selected, if two or more entries to a problem are essentially equal, then preference may be given to the students in the earlier year of university.

Contestants may use any source of information except other people. Solutions are to be received by 4.00 pm on Friday, September 9, 2005. They may be given to Dr. Donald Cartwright, Room 620, Carslaw Building, or posted to him at the School of Mathematics and Statistics, The University of Sydney, N.S.W. 2006. Entries must state name, university, student number, course and year, term address and telephone number, and be marked **2005 SUMS Competition**. The prizes will be awarded towards the end of the academic year.

The SUMS committee is grateful to all those who have provided problems. We are always keen to get more. Send any, with solutions, to Dr. Cartwright, at the above address.

These problems will also be posted at the website

http://www.maths.usyd.edu.au/u/SUMS/

Problems

(Extensions and generalizations of any problem are invited and are taken into account when assessing solutions.)

1. Suppose that we look at the set X_n of strings of 0's and 1's of length n. Given a string $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in X_n$, we are allowed to change ϵ according to the following rules:

(i) We are allowed to change ϵ_1 to $1 - \epsilon_1$;

(ii) if $2 \leq k \leq n$, we are allowed to change ϵ_k to $1 - \epsilon_k$ if and only if $\epsilon_{k-1} = 1$ and $\epsilon_j = 0$ for all $j < k - 1$.

Find a formula for the least number of such changes needed to turn the string $(0, 0, \ldots, 0)$ into $(1, 1, \ldots, 1)$.

2. Determine all positive integers n for which

$$\lfloor 2\sqrt{n} \rfloor = 1 + \lfloor \sqrt{n-1} + \sqrt{n+1} \rfloor.$$

3. Let $P(x)$ be a polynomial with real coefficients such that, for some constants a, b,

$$P(x) + aP'(x) + bP''(x) \geq 0 \quad \text{for all } x \in \mathbb{R}.$$

Suppose that $a^2 \geq 4b$. Show that $P(x) \geq 0$ for all $x \in \mathbb{R}$.

4. Suppose that a, b and k are integers, such that

$$0 < a^2 + b^2 - kab \leq k$$

Prove that $a^2 + b^2 - kab$ is a perfect square.

5. Show that there is no four term arithmetic progression, whose terms are distinct square numbers.

6. Form a graph with vertex set $\mathbb{Z}$ in which there is an edge from m to n if and only if $|m-n| \in \{1, 2, 4, 8, \ldots\}$. If $m, n \in \mathbb{Z}$, let $\operatorname{dist}(m, n)$ denote the distance from m to n in this graph, that is, the length of a shortest path joining m and n. For example, $\operatorname{dist}(0, 6) = 2$ because there is an edge from 0 to 4 and one from 4 to 6, but there is no edge from 0 to 6. Now let $B = \{n \in \mathbb{Z} : \operatorname{dist}(n, 0) \leq r\}$ be the "ball of radius r about 0" in this graph. Show that the complement of B is connected. That is, show that it is possible to join any two points outside B by a path all of whose vertices lie outside B.

7. Define a function $f : \mathbb{R}^3 \to \mathbb{C}$ by

$$f(\theta_1, \theta_2, \theta_3) = e^{i(\theta_1+\theta_2+\theta_3)} + e^{i(\theta_1-\theta_2-\theta_3)} + e^{i(-\theta_1+\theta_2-\theta_3)} + e^{i(-\theta_1-\theta_2+\theta_3)}.$$

Find the image of f.

8. Calculate the determinant of the $n \times n$ matrix A with entries

$$a_{i,j} = \frac{(2i + 2j - 2)!}{2^{2i+2j-2}(i + j - 1)!} \quad (i, j = 1, \ldots, n).$$

9. A square $n \times n$ block of government owned land is divided into n^2 square 1×1 lots. There are n farmers, and initially, each farmer buys one lot, so that each row and each column contains only one privatized lot. On each succeeding season, each farmer may acquire another lot if it is not yet privatized and if it has a common side with the lot privatized by this farmer the previous season.

For which n is it possible to privatize n lots initially in such a way that the whole block can be privatized in n seasons?

10. To 'triangulate' a convex polygon means to draw straight lines between pairs of non-adjacent vertices, such that no two lines intersect in the interior of the polygon, and the interior is thereby subdivided into triangles. It is well known that the number of triangulations of a convex n-gon is the Catalan number c_{n-2}, where c_n is defined by the recursion $c_n = c_0 c_{n-1} + c_1 c_{n-2} + \cdots + c_{n-1} c_0$, with initial value $c_0 = 1$.

Now suppose that P is a regular n-gon. We say that two triangulations are equivalent if one can be obtained from the other by rotating or reflecting P. Find a formula for the number of equivalence classes of triangulations, in terms of the Catalan numbers.