The Calderón Problem -
From the Past to the Present

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FINLANDS AKADEMI

National Science Foundation
WHERE DISCOVERIES BEGIN

Vetenskapsrådet

Part I - The Classical Problem on $\mathbb{R}^n$

1. Calderón's Impedance Tomography Problem

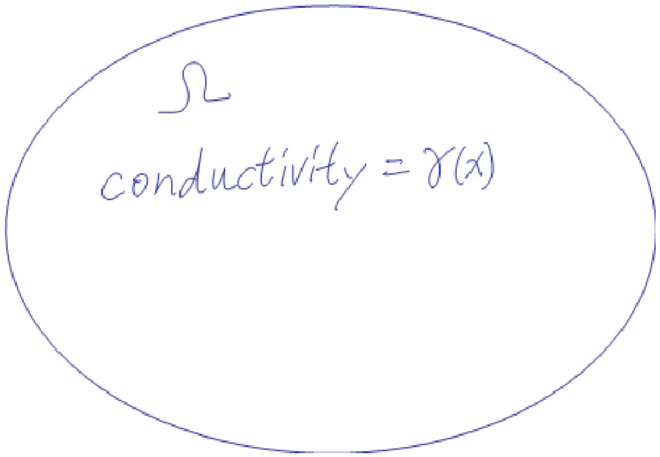
2. Anisotropic Medium and Non-uniqueness

3. Sylvester-Uhlmann Solution for Isotropic Medium
    - Boundary Integral Identity
    - Complex Geometric Optics

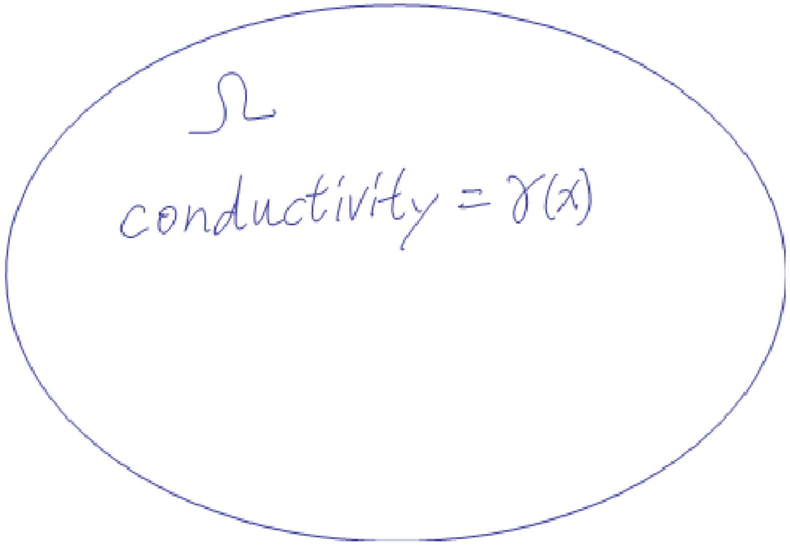
Part II - The Manifold Setting

1. Geometric Aspects of PDE

2. Some Geometric Techniques



- Material Ω with conductivity $\gamma(x)$
- In general the material is anisotropic (muscle, timber, etc.)
- Conductivity depends on direction
- $\gamma(x)$ an $n \times n$ positive definite matrix
- Special isotropic cases (water, breast tissue), $\gamma(x) = \underbrace{\gamma(x)}_{scalar} I_{n\times n}$



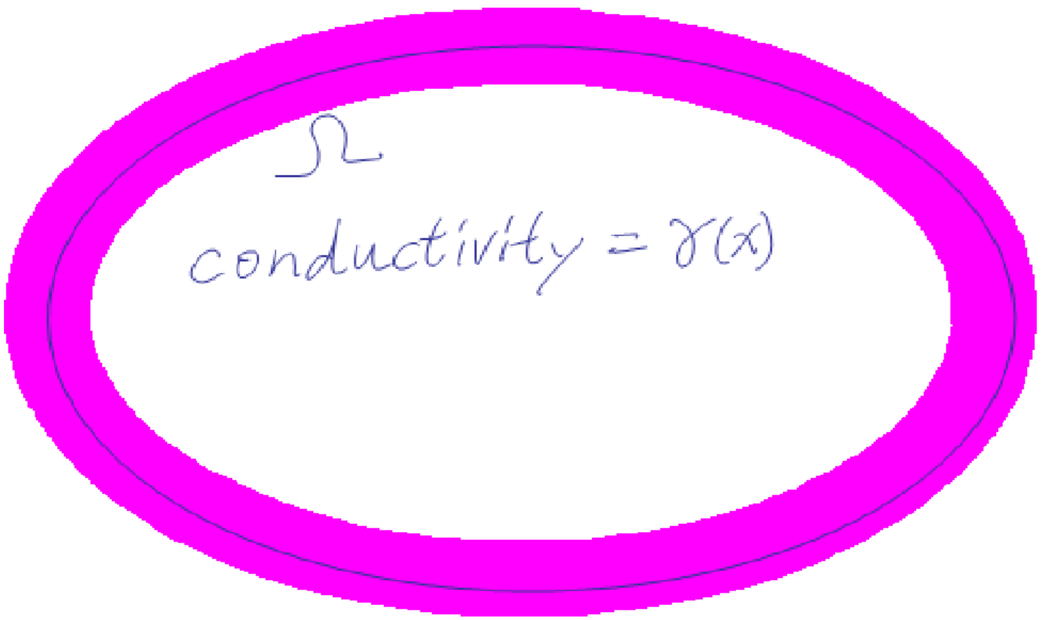
How do we determine $\gamma(x)$ in a non-invasive way?

This question is relevant in:

- Breast tumour detection
- Detecting impurities in steel
- Gas/oil exploration

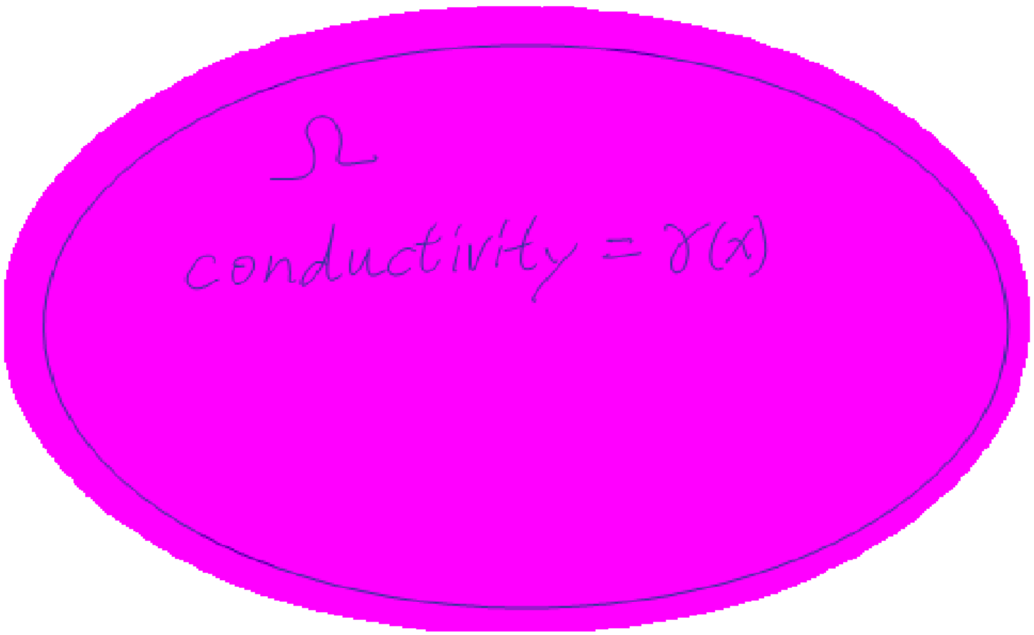
Electric Impedance Tomography (EIT):

We apply a voltage on the boundary.



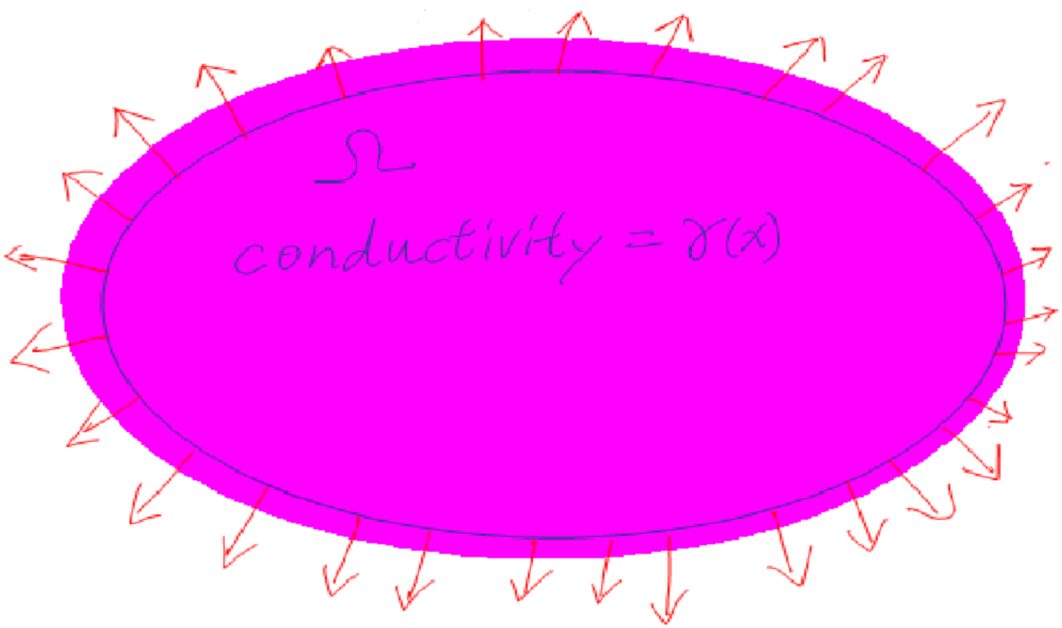
Electric Impedance Tomography (EIT):

This surface voltage induces an internal voltage.



Electric Impedance Tomography (EIT):

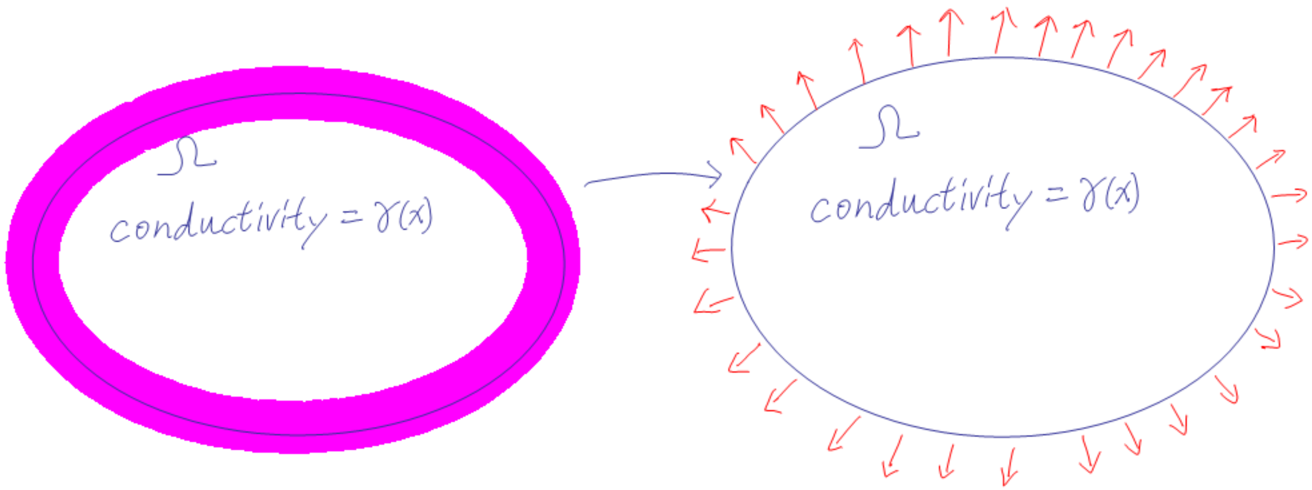
The voltage then gives a surface electric flux (current)



which we can measure.

Electric Impedance Tomography (EIT):

The lab technician can only measure what happens on the outside.



and record the resulting data:

Input Voltage	f_1	f_2	f_3	etc...
Output Current	c_1	c_2	c_3	etc...

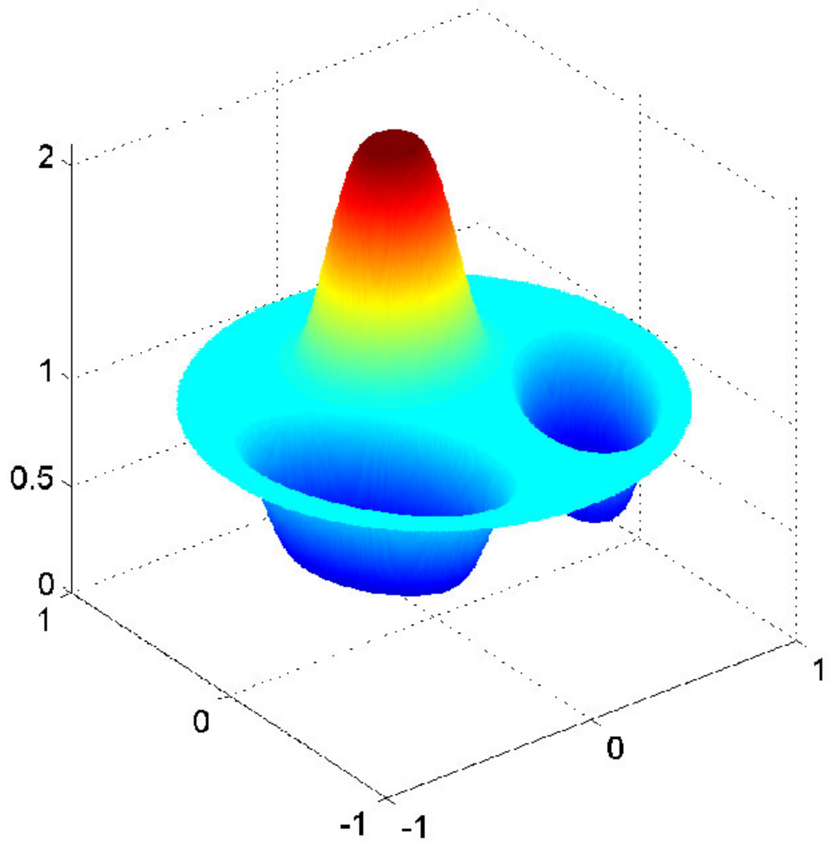
Electric Impedance Tomography (EIT):

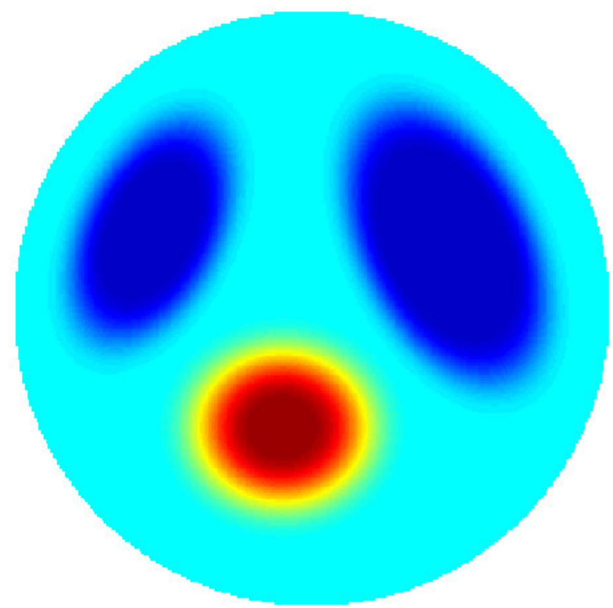
- The data depend on the conductivity γ.
- From the recorded data we recover the conductivity

A real life experiment. Data collected with 32 electrodes:

The machine is in Rensselaer Polytechnic Institute, USA.

Numerical reconstruction from data:





Courtesy of Dr. Siltanen of Finnish Centre of Excellence in Inverse Problems Research

- The pictures look reasonable but....

- Two different conductivities could potentially give identical measurements.

- Need to prove that this doesn't happen.

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- Dependence of Λ_γ on γ NONLINEAR.

Calderón's Problem:

Does the operator Λ_γ uniquely determine γ?
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For general anisotropic (matrix valued) γ the answer is NO.

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- Let $F : \Omega \to \Omega$ be a diffeomorphism with $F\mid_{\partial\Omega} = Id$.

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- Intuition from differential geometry

- Harmonic functions are invariant under pull-back by isometries.

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Is this the only non-uniqueness?

Conjecture
Suppose $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$. Then there exists a diffeomorphism

$$F : \Omega \to \Omega, \quad F\,|_{\partial\Omega} = Id$$

such that $\gamma_2 = F_* \gamma_1$.

- Only known to be true if $\Omega \subset \mathbb{R}^2$ (Nachman, Sylvester, Astala-Lassas-Päivärinta).

- $n \geq 3$ open.

Isotropic Conductivities

Now suppose a-priori that γ is isotropic (a scalar function).

Theorem (Sylvester-Uhlmann)
Let $\Omega \subset \mathbb{R}^n$ for $n \geq 3$. Suppose γ_1 and γ_2 are two smooth scalar conductivities such that

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2},$$

then $\gamma_1 = \gamma_2$.

- Non-constant coefficient $\nabla \cdot \gamma \nabla$ is not so nice.
- The proof considers an auxiliary problem for the Schrödinger operator $\Delta + V$.

Schrödinger Operator $\Delta + V$ and its Dirichlet-Neumann map

- Let $V \in L^\infty(\Omega)$ be the potential.

- Assume for all $f \in C^\infty(\partial\Omega)$, $\exists! u_f$ solving

$$(\Delta + V)u_f = 0 \ on \ \Omega$$

$$u_f = f \ on \ \partial\Omega$$

- Define Dirichlet-Neumann map $\Lambda_V : C^\infty(\partial\Omega) \to C^\infty(\partial\Omega)$ by

$$\Lambda_V : f \longmapsto \partial_\nu u_f$$

- $\Lambda_{V_1} = \Lambda_{V_2} \implies V_1 = V_2$? Yes
  ($n \geq 3$ Sylvester-Uhlmann, $n = 2$ Bukgheim)

- For isotropic conductivity, $\nabla \cdot \gamma \nabla$ is a special case of $\Delta + V$

- Take $V = \frac{-\Delta\sqrt{\gamma}}{\sqrt{\gamma}}$ and make a change of variable.

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- Prove: $\Lambda_{V_1} = \Lambda_{V_2} \implies V_1 = V_2$.

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2. Probe identity with special solutions.

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Will show products of solutions "look like" Fourier Transforms.

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Construct "Complex Geometric Optics"

2. Probing Identity With Special Solutions

- Recall Fourier Transform of a function:

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Plug

$$u_1 \bar{u}_2 = e^{i\xi \cdot x} + r$$

into

$$\int_\Omega u_1 (V_1 - V_2) \overline{u_2} = 0$$

we have

$$\int_\Omega e^{i\xi \cdot x} (V_1 - V_2) = 0$$

Caveats

- This idea needs $n \geq 3$
- Choice of $\zeta \in \mathbb{C}^n$ requires THREE mutually perpendicular vectors in $\mathbb{R}^n$.
- Idea only works on flat space.

Part II - The Manifold Setting

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First talk about geometry

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First talk about geometry then analysis.

Dimensions $n = 2$

Theorem(Guillarmou - LT, Duke Math J 2011)
Let M be a Riemann surface with boundary. Suppose $V_1, V_2 \in C^\infty(\overline{M})$ satisfy $\Lambda_{V_1} f = \Lambda_{V_2} f$, $\forall f$, then $V_1 = V_2$.

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- $M = M' \times [0, 1]$, $g = \begin{pmatrix} 1 & 0 \\ 0 & g'(x') \end{pmatrix}$

- (M', g') a simple manifold

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Ferreira-Kurylev-Lassas-Salo recently relaxed the assumption on M'.

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In $n = 2$ we can do even better.

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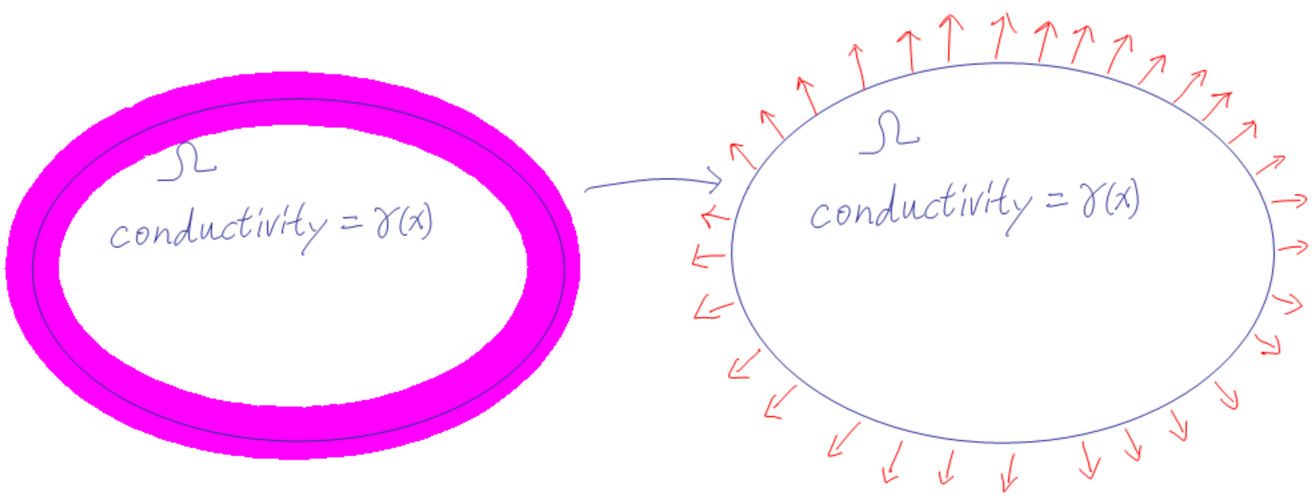
So far we have been able to make measurements on the entire boundary.



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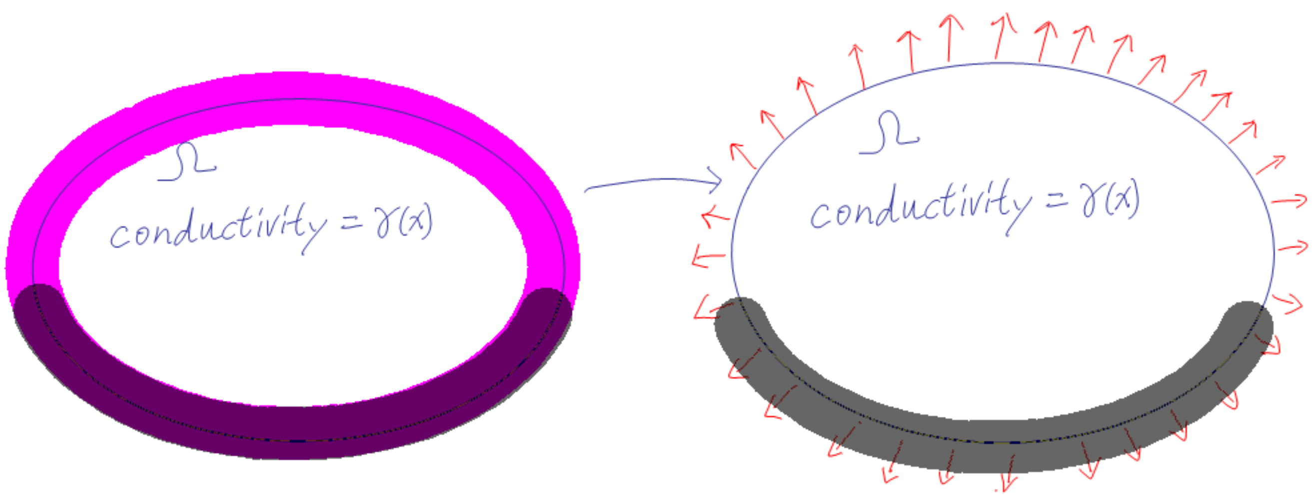
What if part of the boundary is inaccessible?



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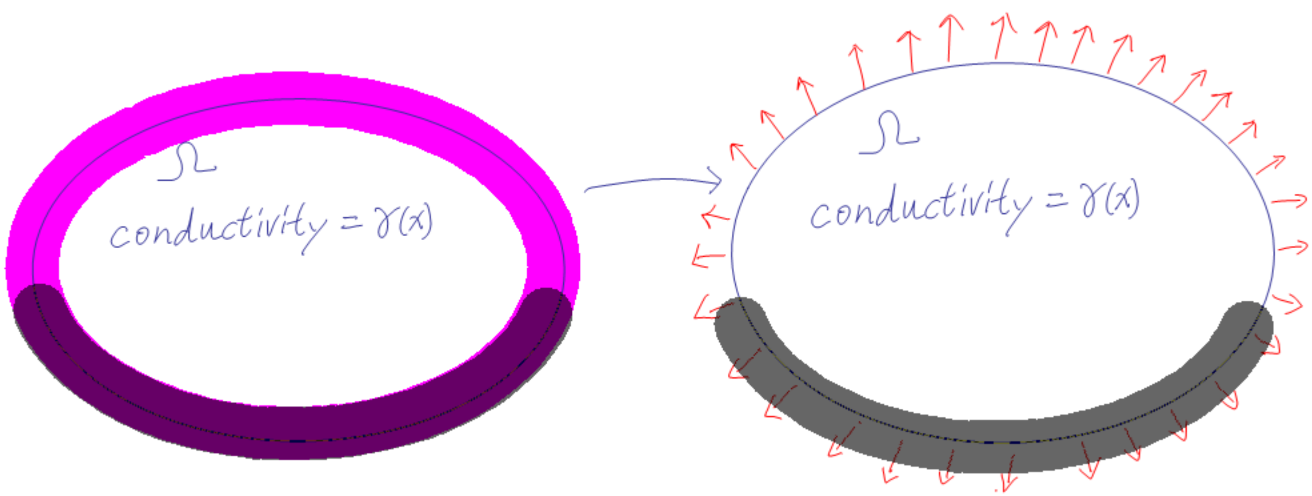
Can only measure on $\Gamma \subset \partial M$ small open subset.



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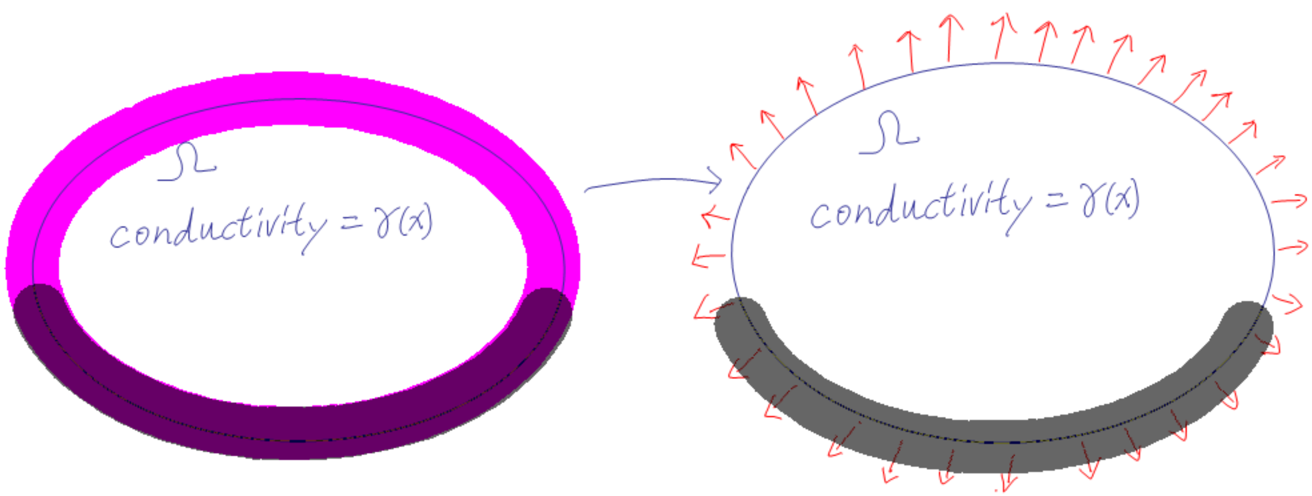

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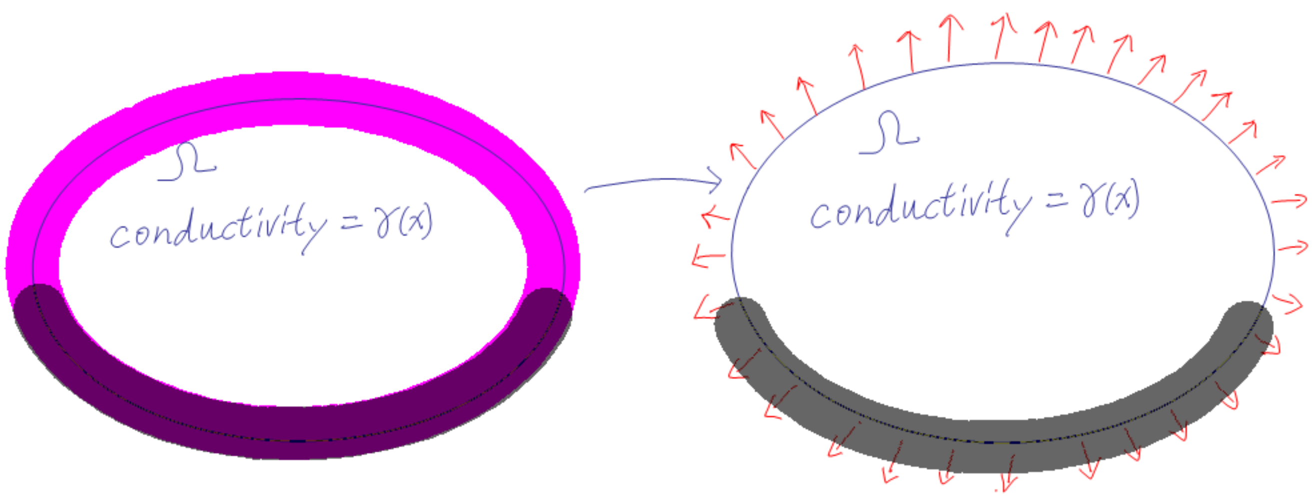

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So far we recovered V from the DN map for the operator

$$d^* d + V.$$

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A a real valued 1-form.

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Connection Laplacian on complex line bundle $E = C \times M$

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What information does its DN map $\Lambda_{A,V}$ give about A and V?

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$$\int_\gamma A \quad \text{mod } 2\pi\mathbb{Z}$$

for all closed curves γ.

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Let M be a Riemann surface with boundary. Suppose $V_1, V_2 \in C^\infty(\overline{M})$ satisfy $\Lambda_{V_1} f|_\Gamma = \Lambda_{V_2} f|_\Gamma$, $\forall f \in C_0^\infty(\Gamma)$, then $V_1 = V_2$.

Theorem(Guillarmou - LT, GAFA 2011)
The DN map of

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Further generalization

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The answer is in the geometry of connection.

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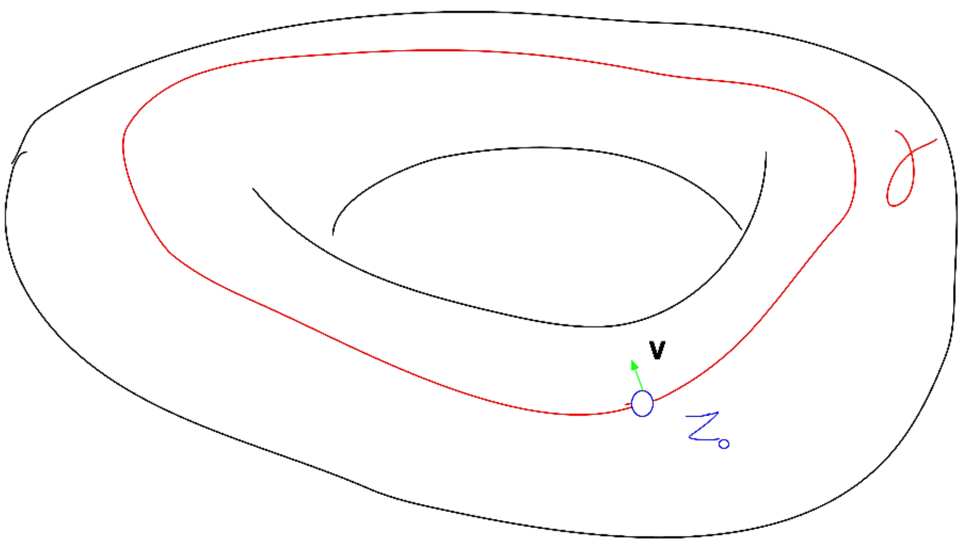

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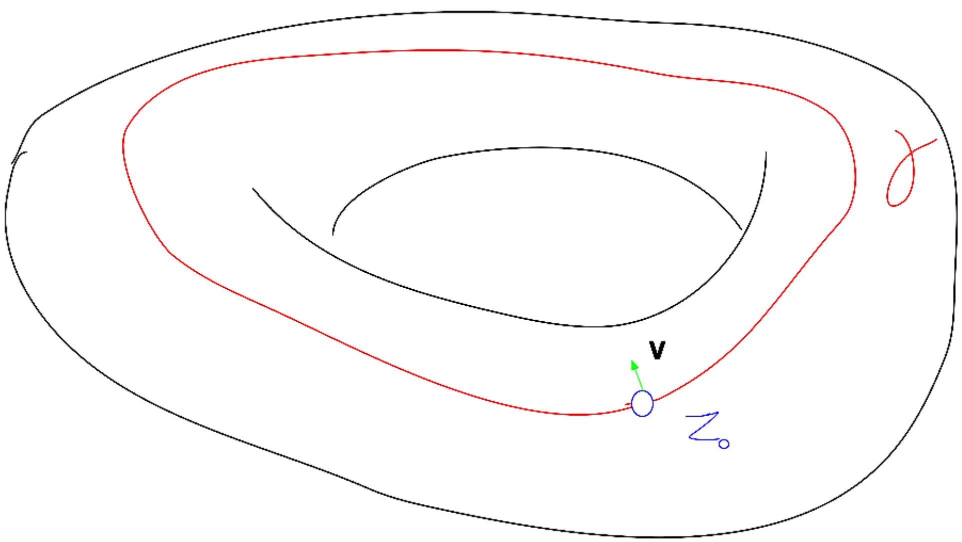

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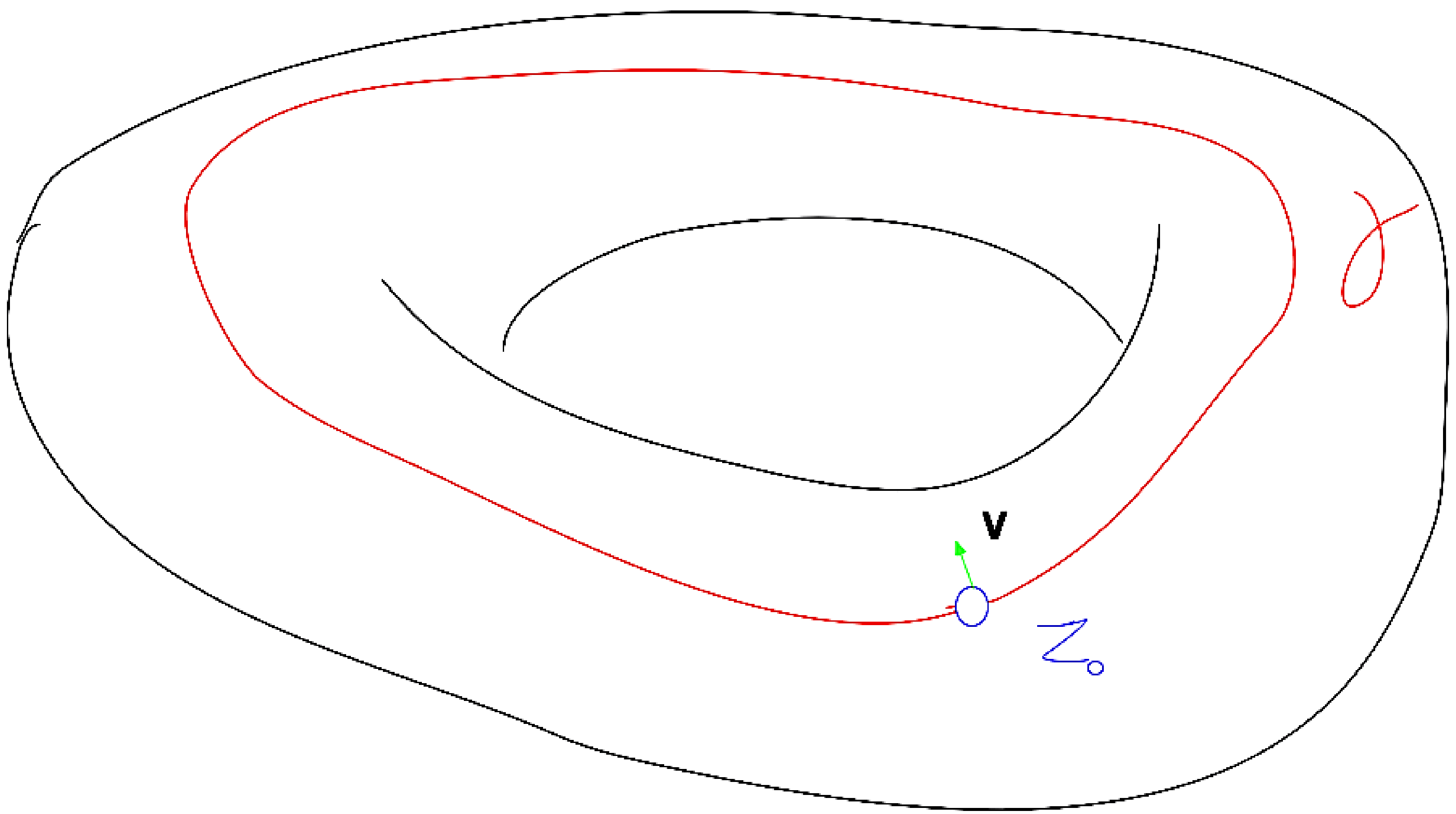

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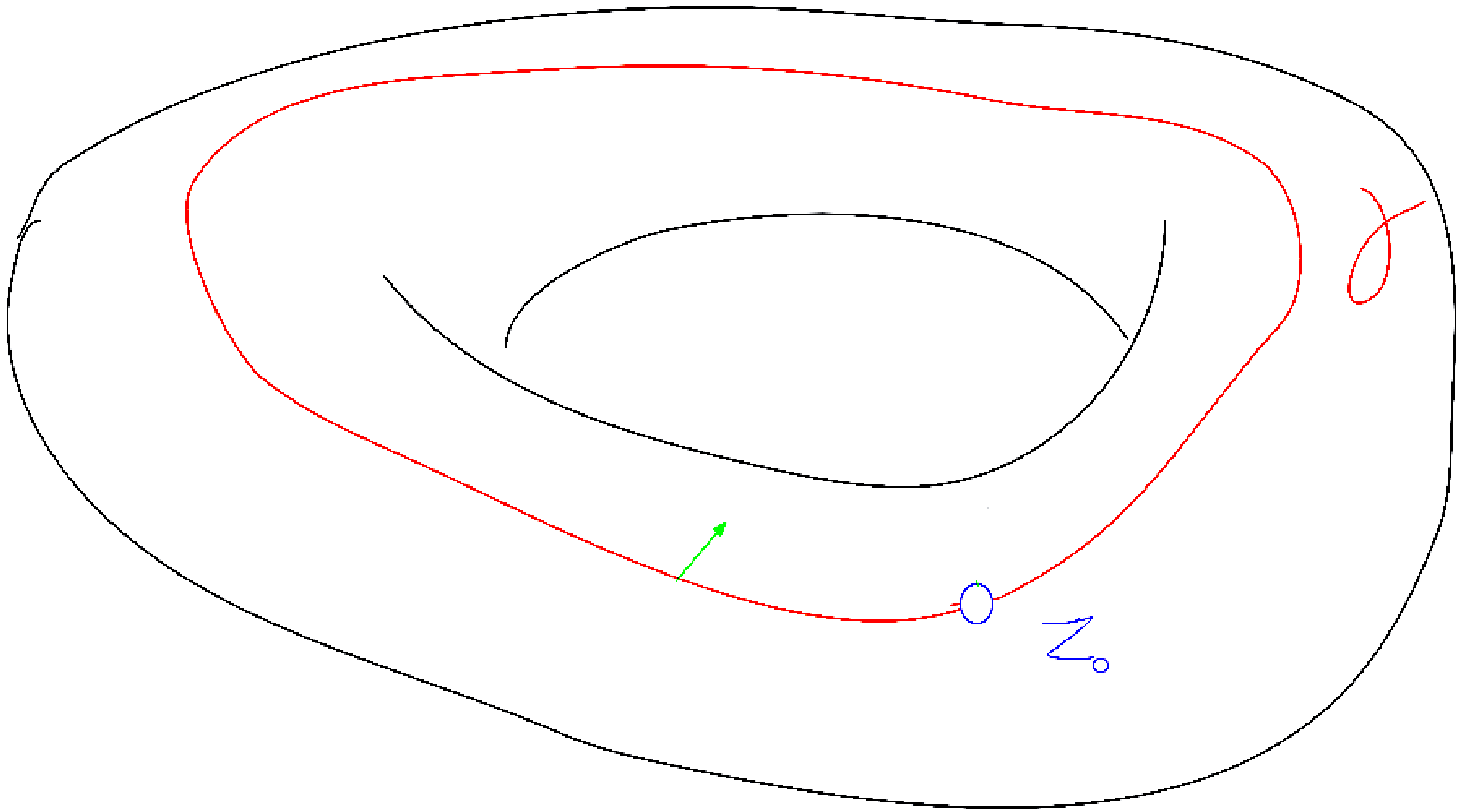

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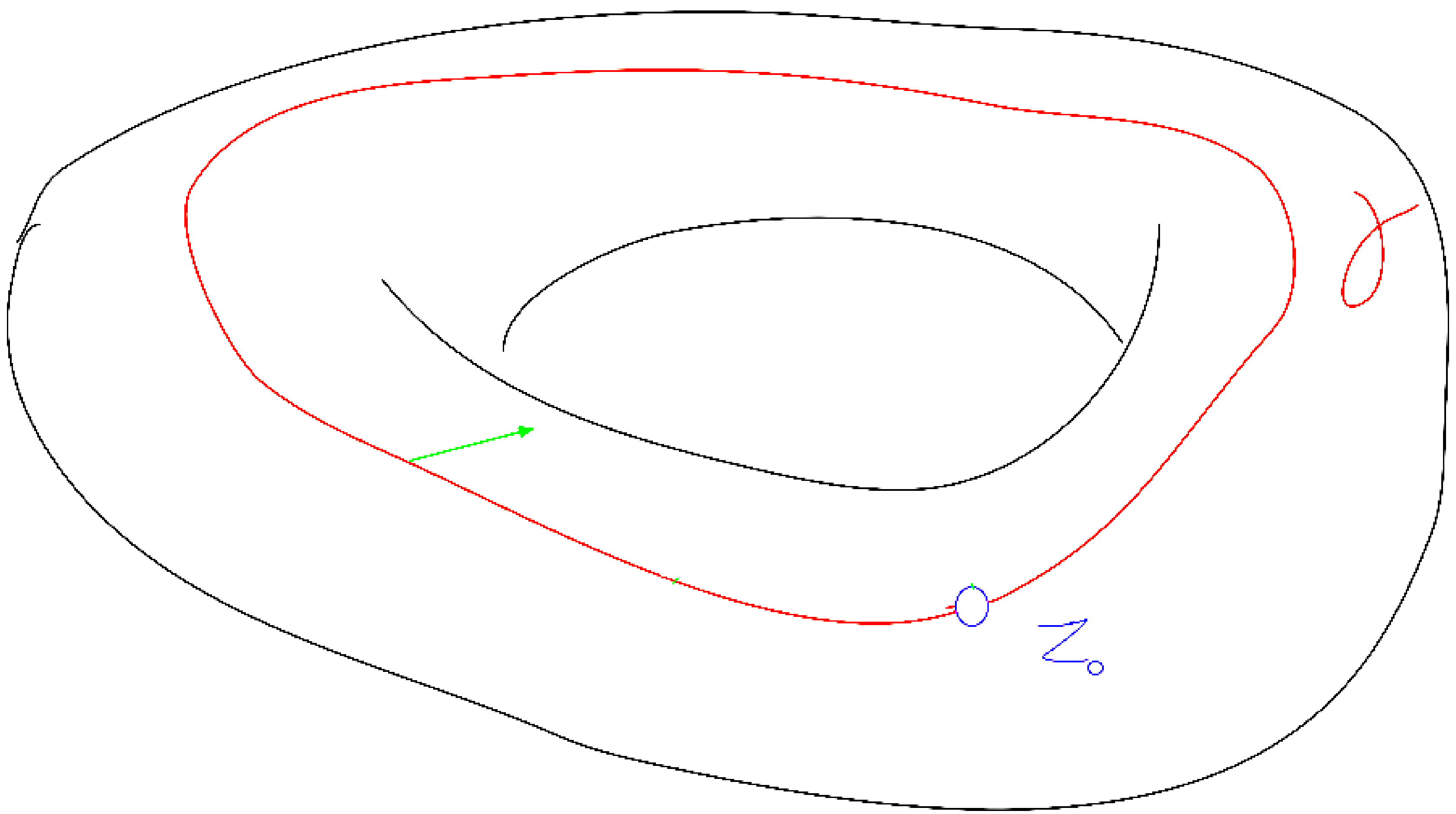

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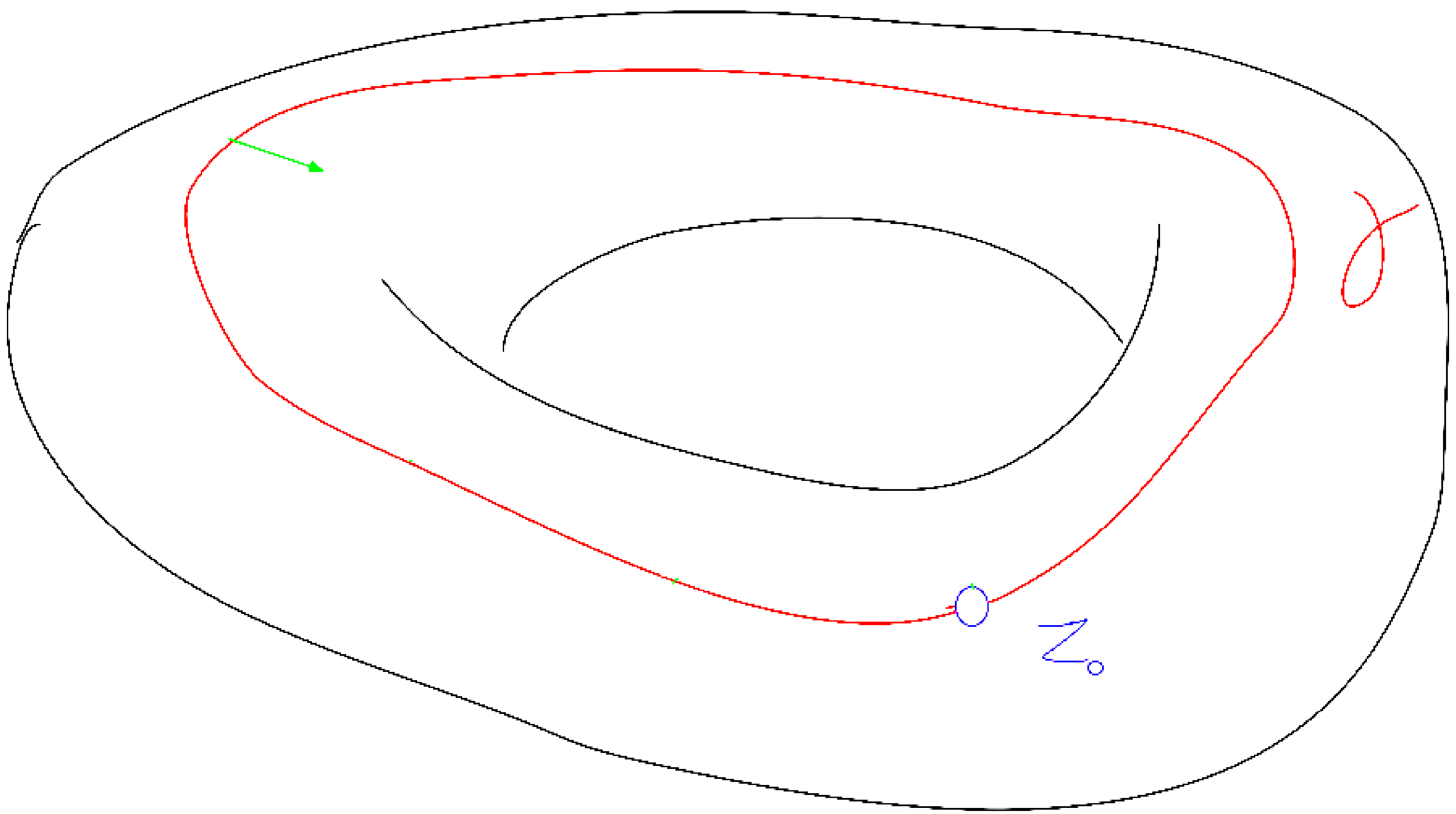

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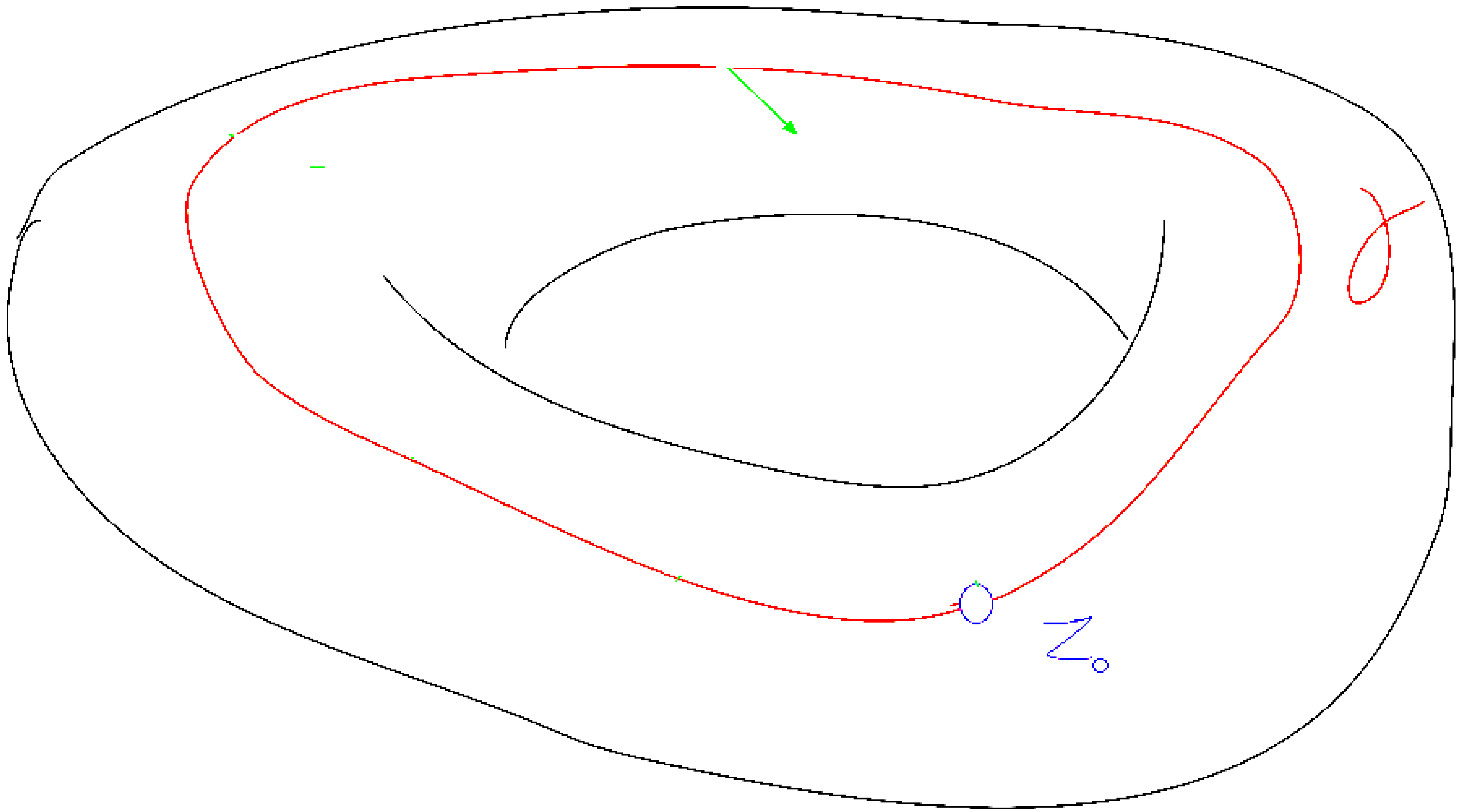

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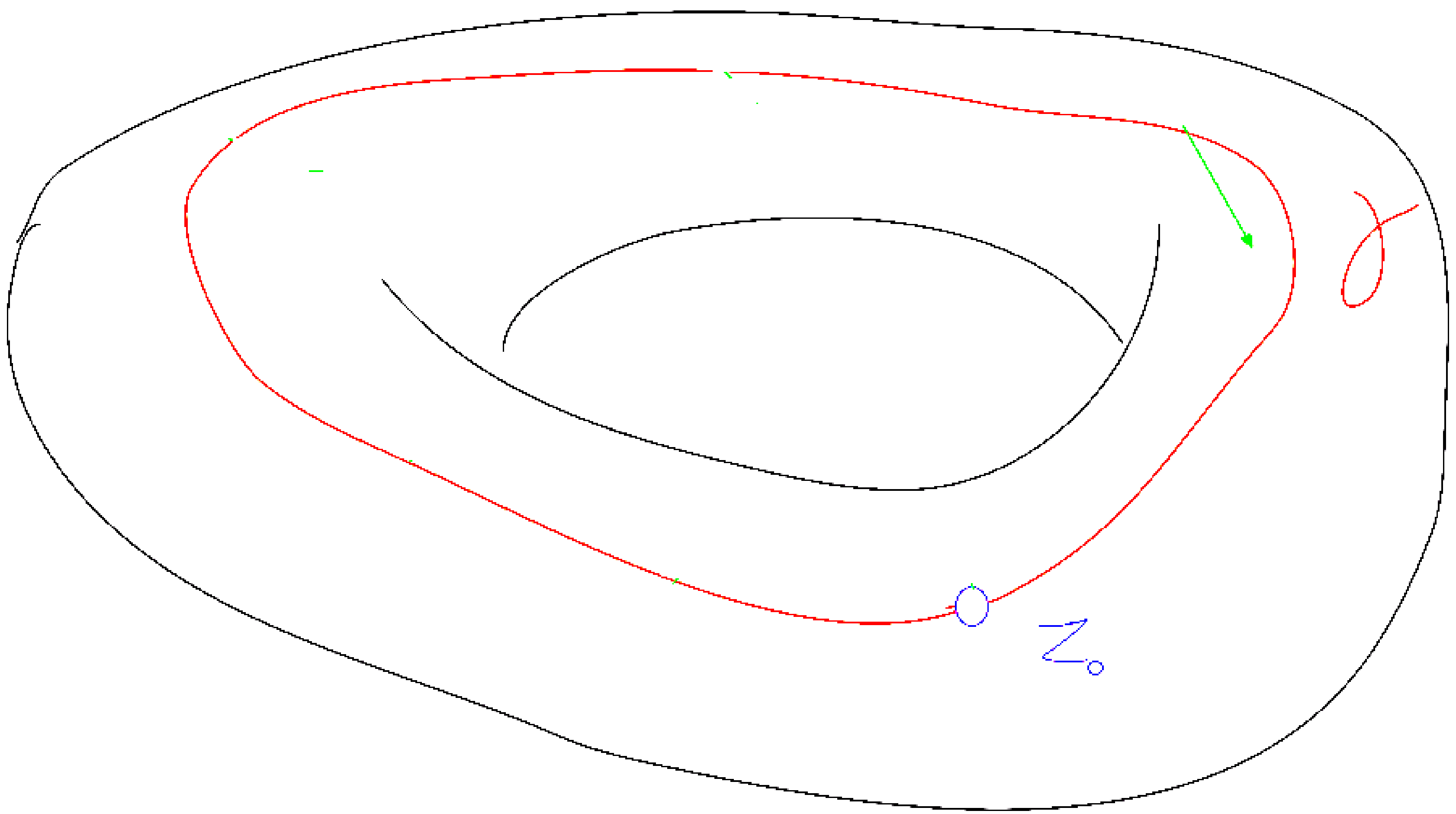

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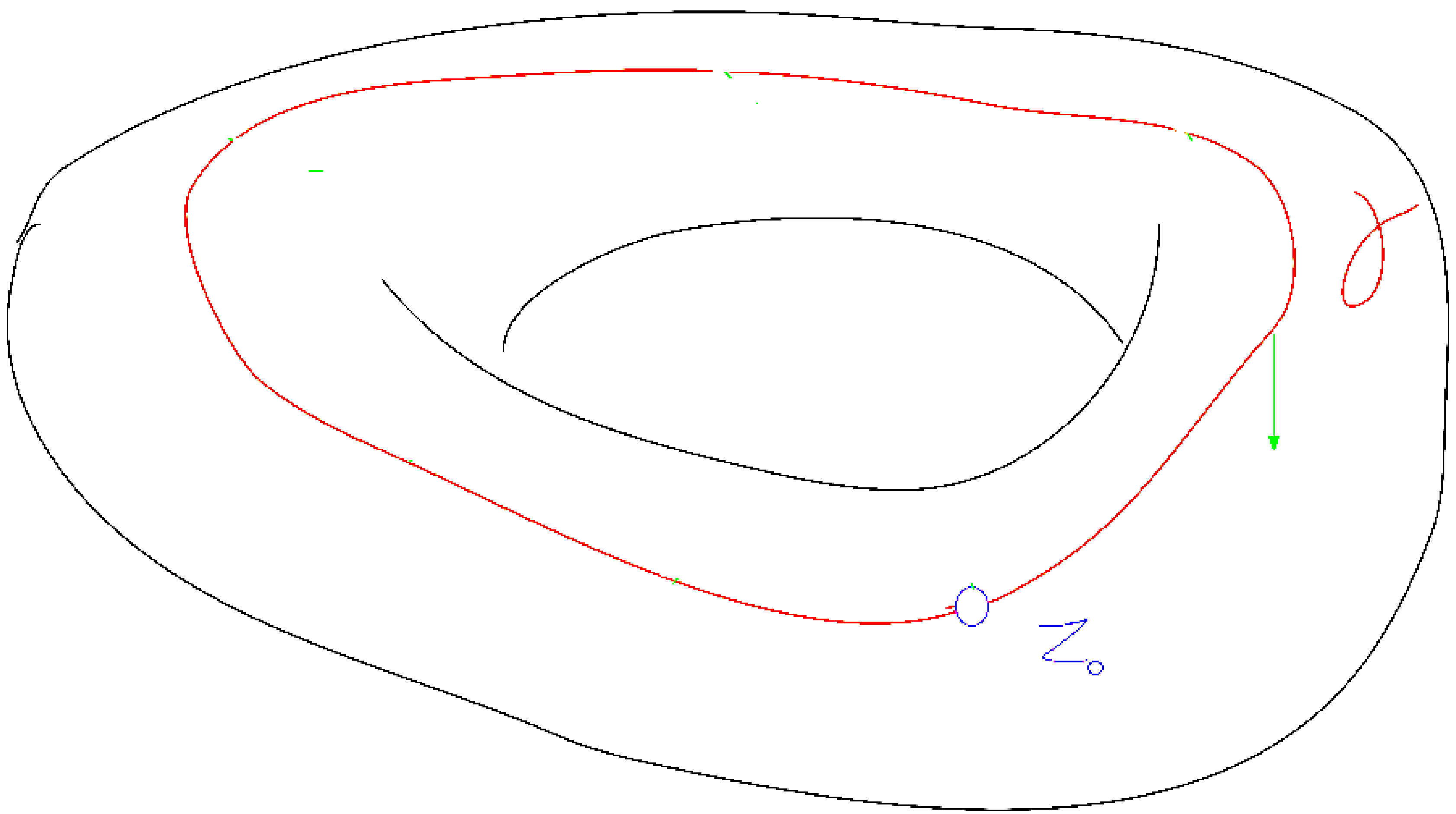

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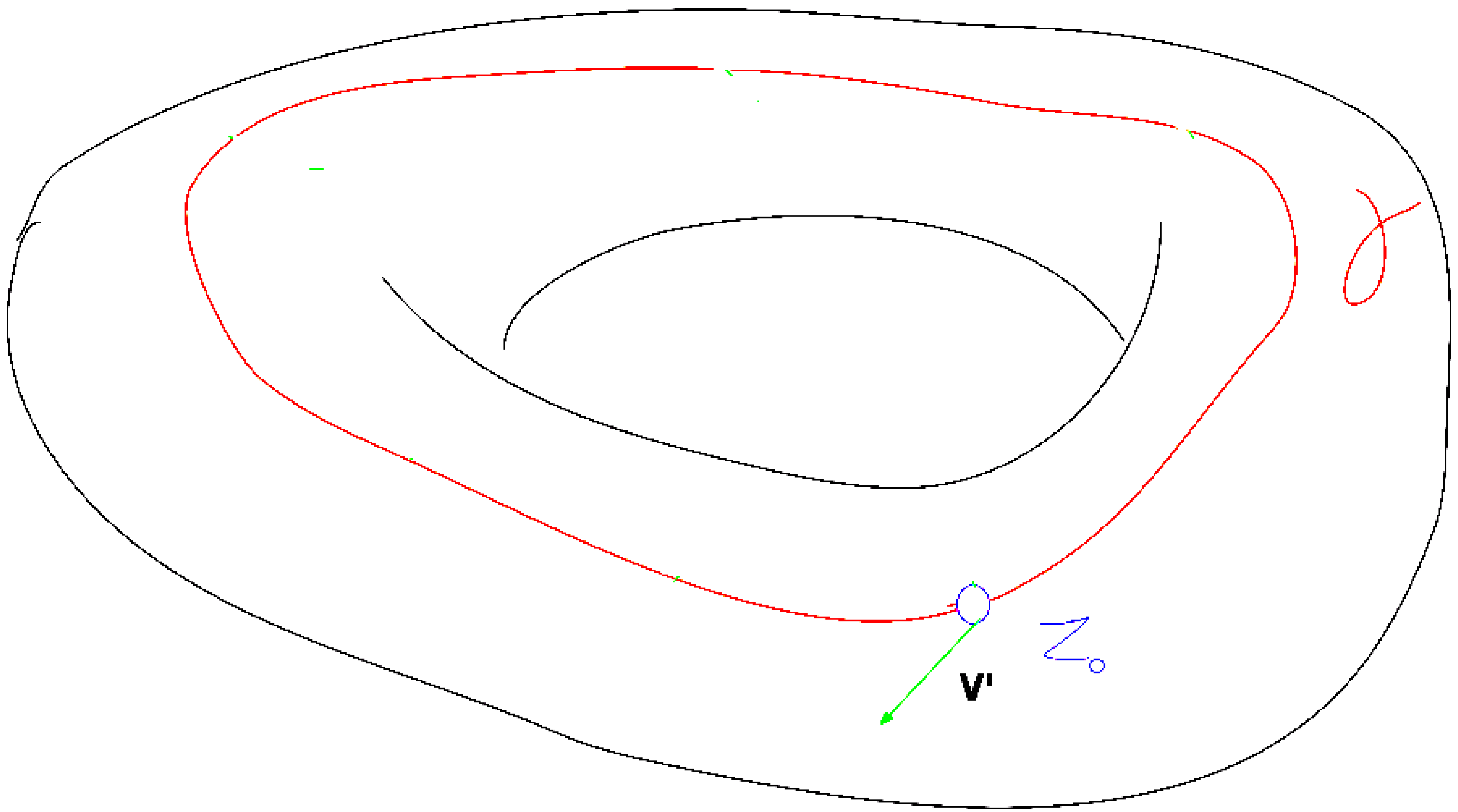

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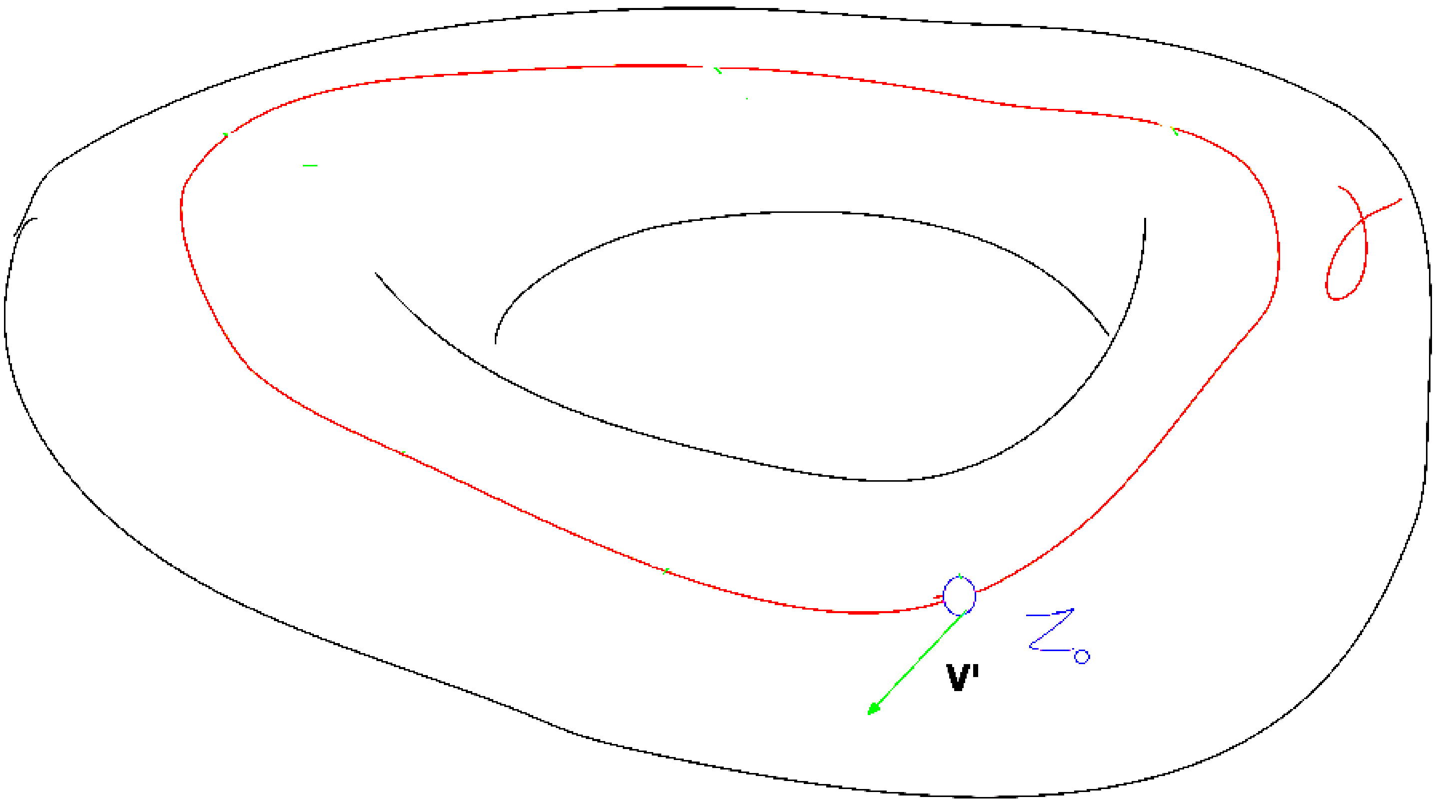

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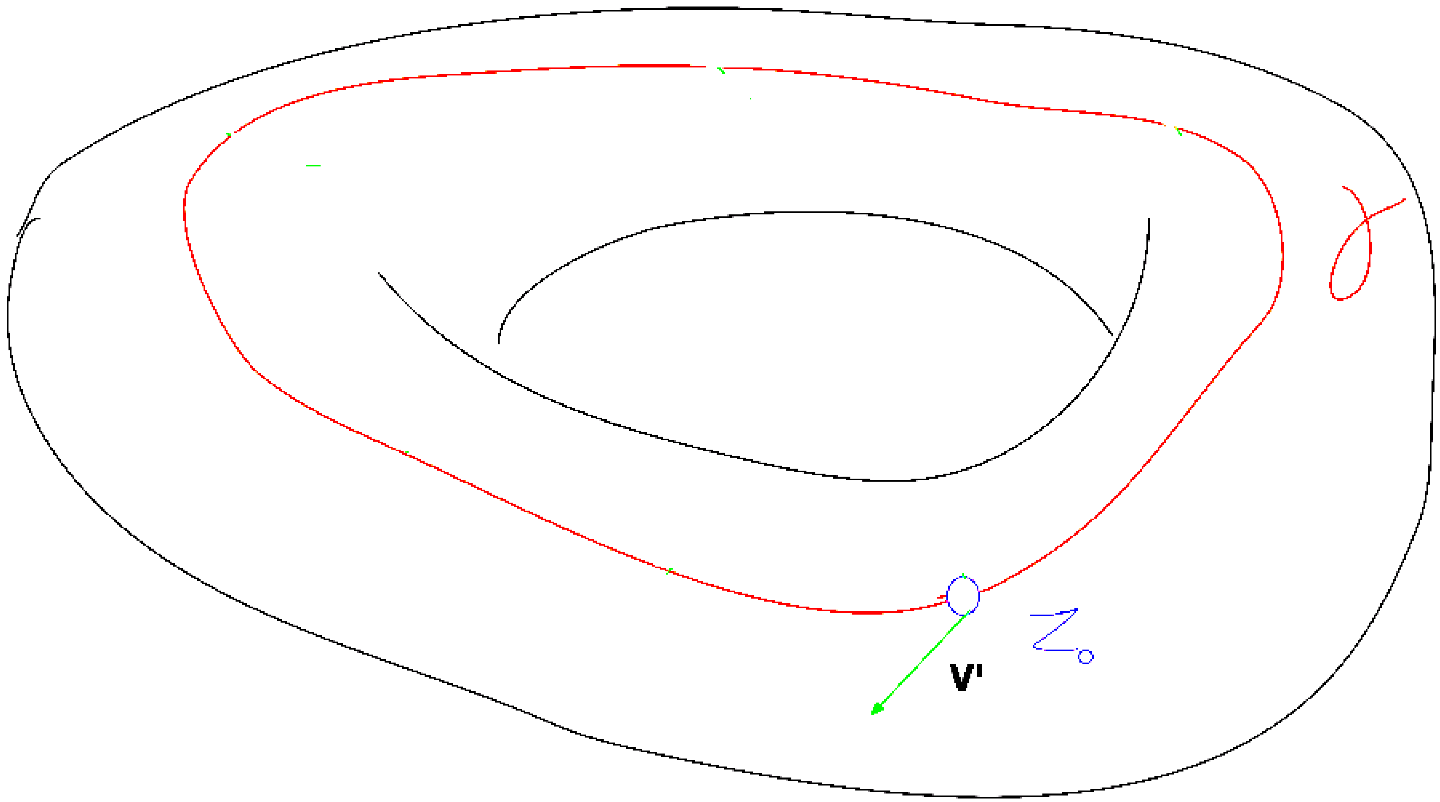

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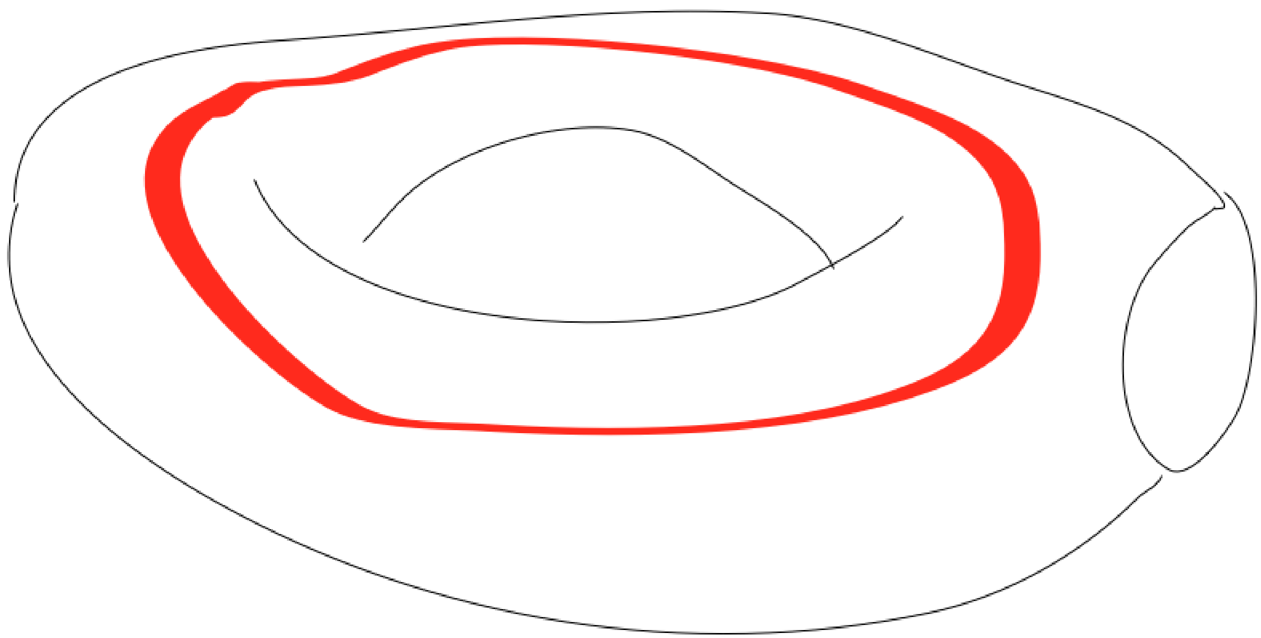
- Geometric intuition of our result.

Proof of Result

Consider a closed loop γ on M:



Want to show that $\int_\gamma A \in 2\pi\mathbb{Z}$.

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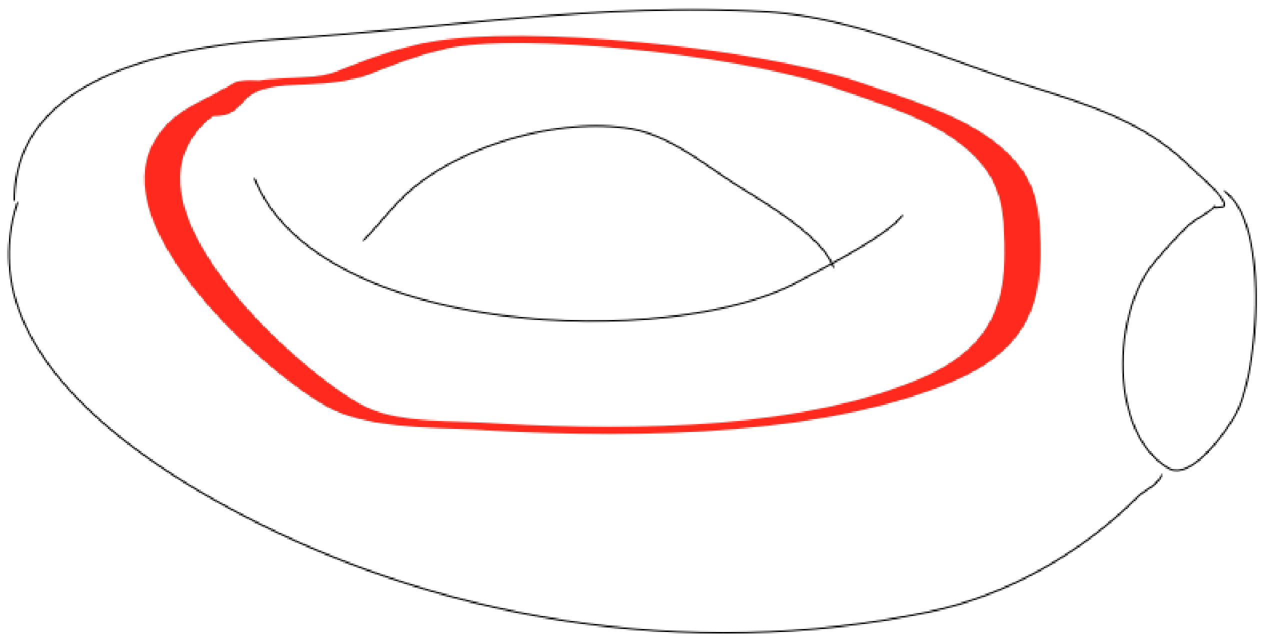

Since $dA = 0$ we can choose any representative of the homology class.

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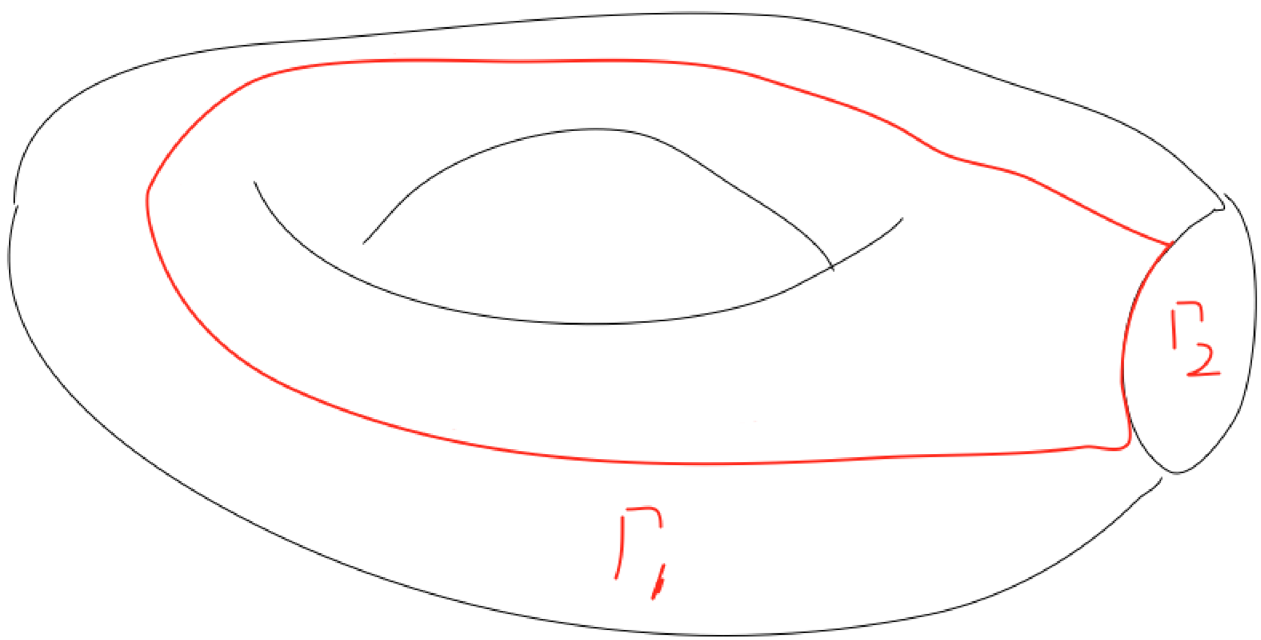

So we deform the curve as such so that part of it, Γ_2, is on ∂M

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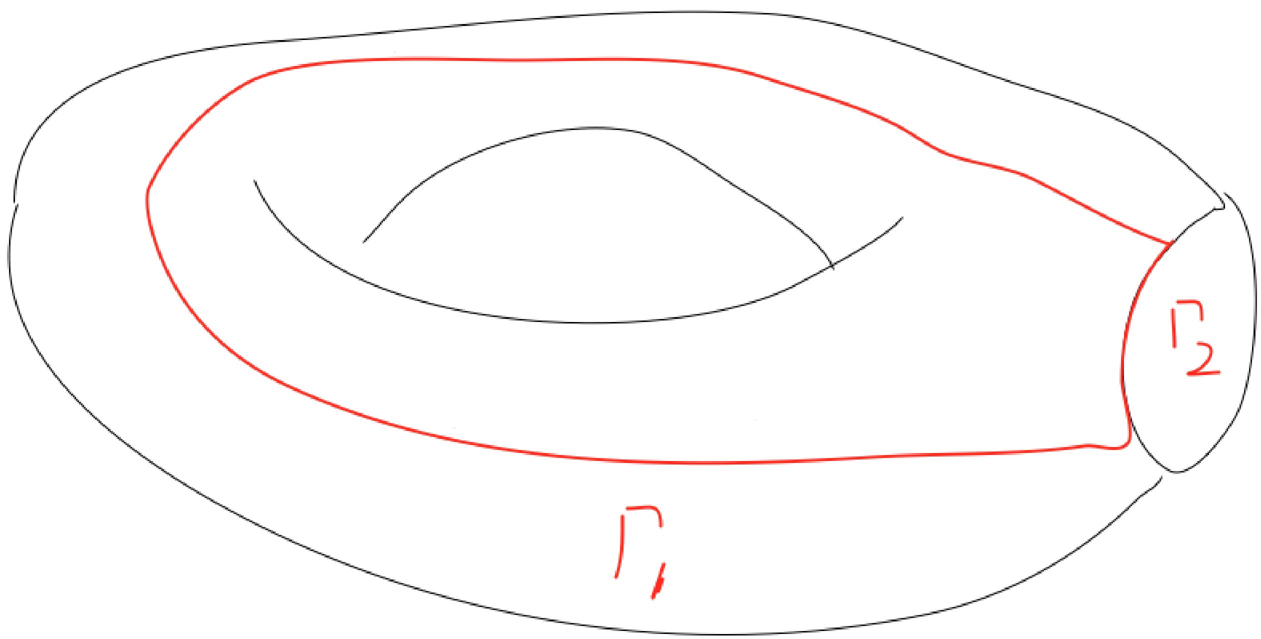

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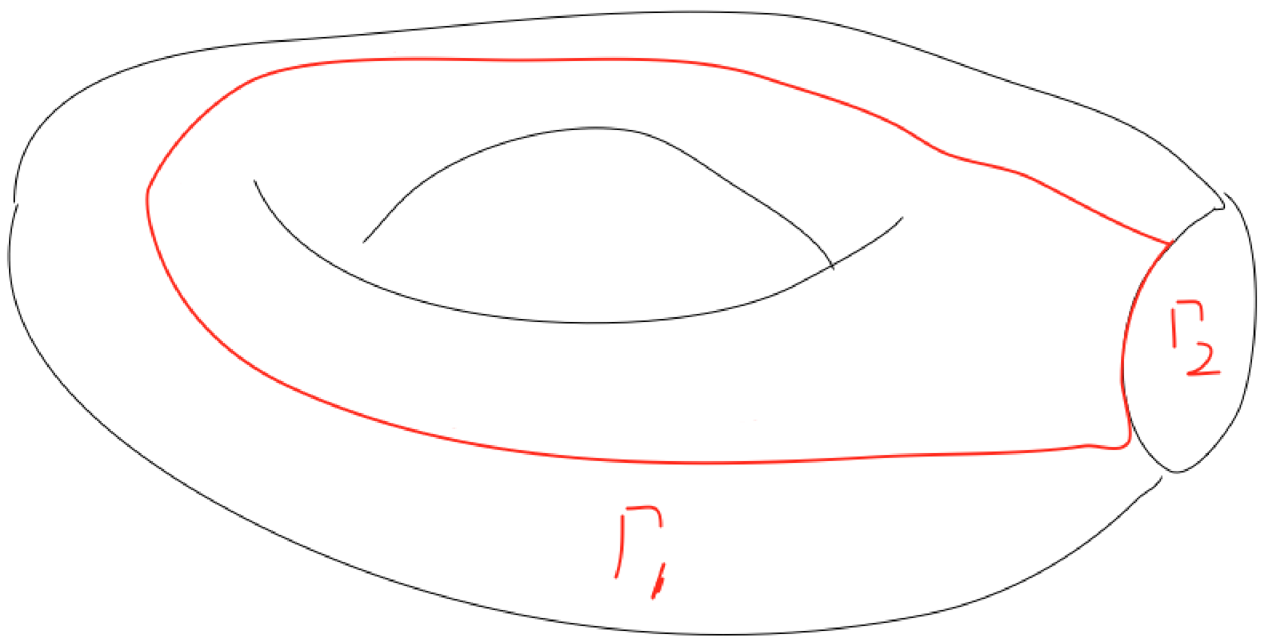

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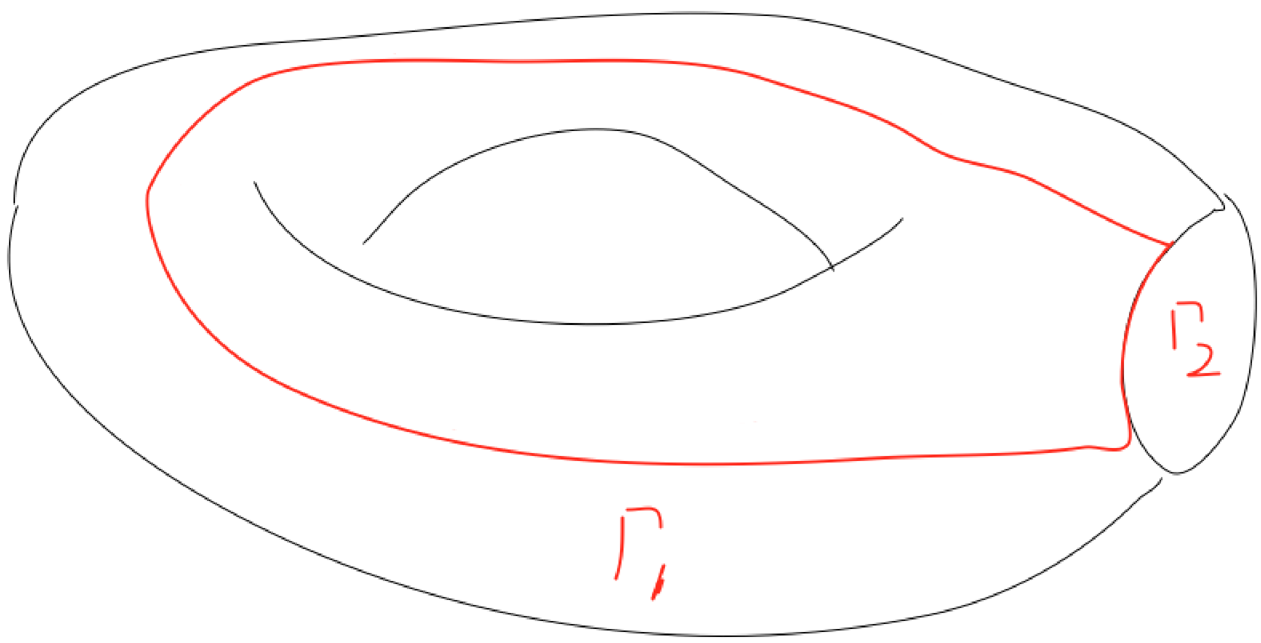
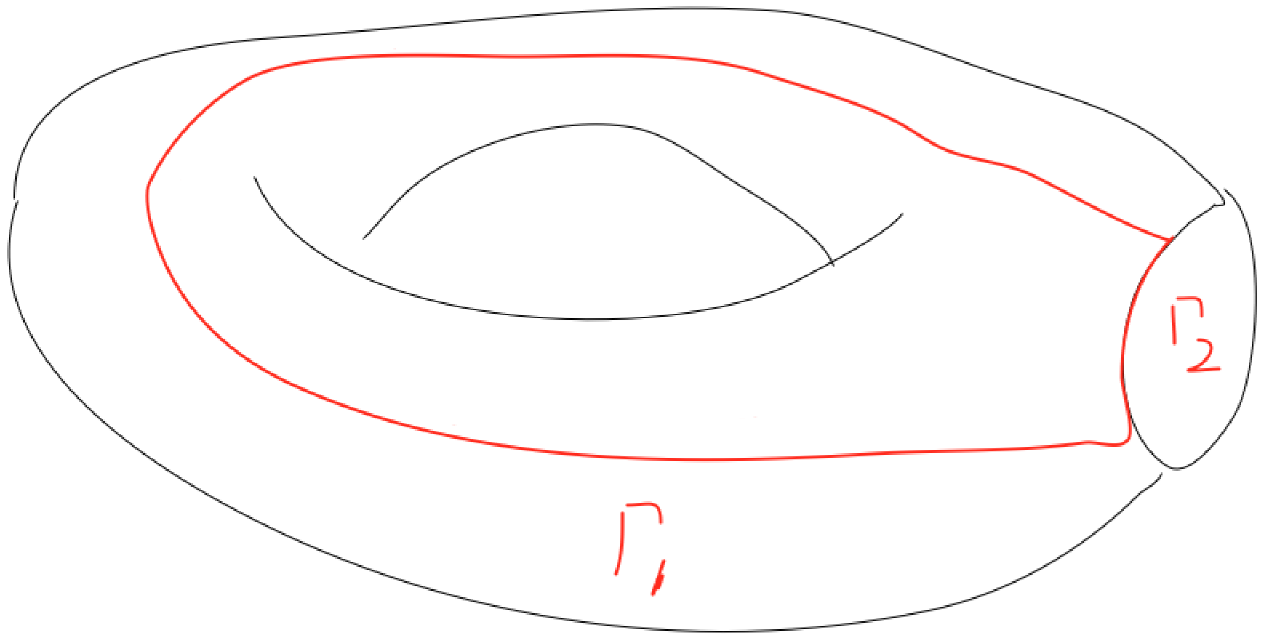


This allows us to propagate information along Γ_1

Proof of Result

Consider a closed loop γ on M:



This allows us to propagate information along Γ_1 QED.

Theorem(Albin - Guillarmou - LT, Ann Henri Poincaré 2013)
Let $\pi : E \to M$ be a Hermitian bundle over surface M and ∇ a Hermitian connection acting on E. Then the DN map of the connection Laplacian

$$\nabla^* \nabla + V$$

determines V and ∇ up to unitary equivalence.

Cauchy-Riemann Operator and Holomorphic Structure

We start with a connection on complex bundle E:

∇

Cauchy-Riemann Operator and Holomorphic Structure

Which determines a Cauchy-Riemann operator:
$\nabla \to \pi_{1,0}\nabla := \partial^{\nabla}$

Cauchy-Riemann Operator and Holomorphic Structure

Which induces a compatible holomorphic structure on E (Kobayashi):
$\nabla \to \pi_{1,0}\nabla := \partial^{\nabla} \to (\mathcal{U}_\alpha, \phi_\alpha)$

Cauchy-Riemann Operator and Holomorphic Structure

Since M has boundary E has a holomorphic trivialization F:
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Play this game for ∇^j, $j = 1, 2$, we get holomorphic trivializations F_1 and F_2 respectively.

Having the Dirichlet-Neumann map of ∇^1 and ∇^2 agree means we can choose holomorphic trivializations F_1 and F_2 such that they agree on ∂M.

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- On M how do we show density of products of solutions?
- In $\mathbb{R}^n$ use Fourier Transform.

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$$\int_{\mathbb{D}} \underbrace{e^{i\psi/h}(V_1 - V_2)}_{principal\ \ part} + o(h) = 0$$

5. $\psi(x, y) = xy$ has a unique non-degenerate critical point at 0.

$$\underbrace{\int_{\mathbb{D}} e^{i\psi/h}(V_1 - V_2)}_{h(V_1 - V_2)(0) + o(h)} + o(h) = 0$$

by stationary phase.

6. $V_1(0) = V_2(0)$. But there is nothing special about the origin. We can put critical point anywhere we like.

General Surfaces

Theorem(Guillarmou - LT, Duke Math J 2011)
Let M be a Riemann surface with boundary. Suppose $V_1, V_2 \in C^\infty(\overline{M})$ satisfy $\Lambda_{V_1} f = \Lambda_{V_2} f$, $\forall f$, then $V_1 = V_2$.

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In $n = 2$ we can do even better.

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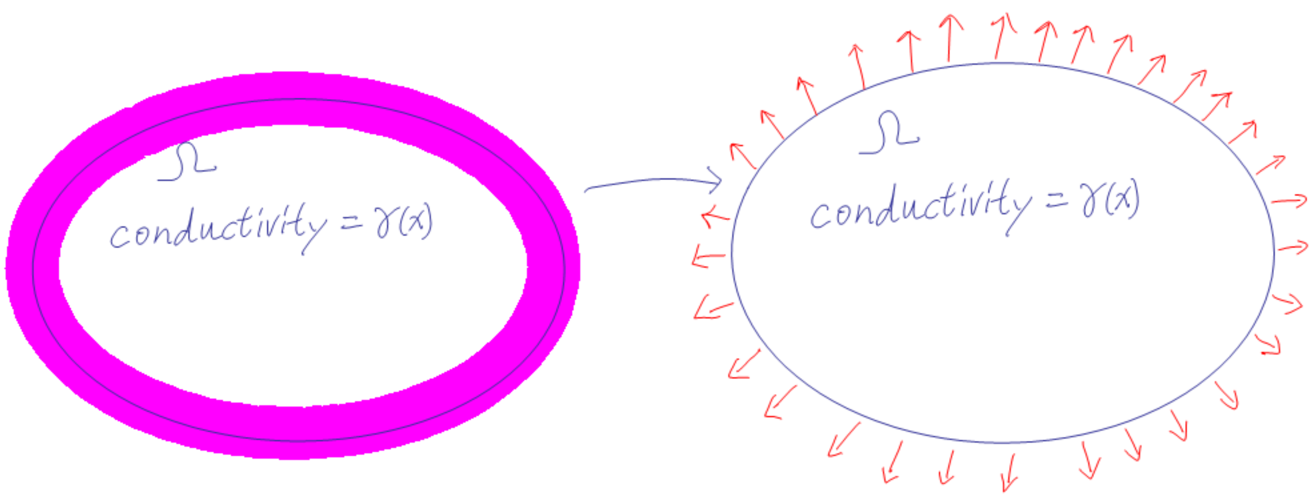
So far we have been able to make measurements on the entire boundary.



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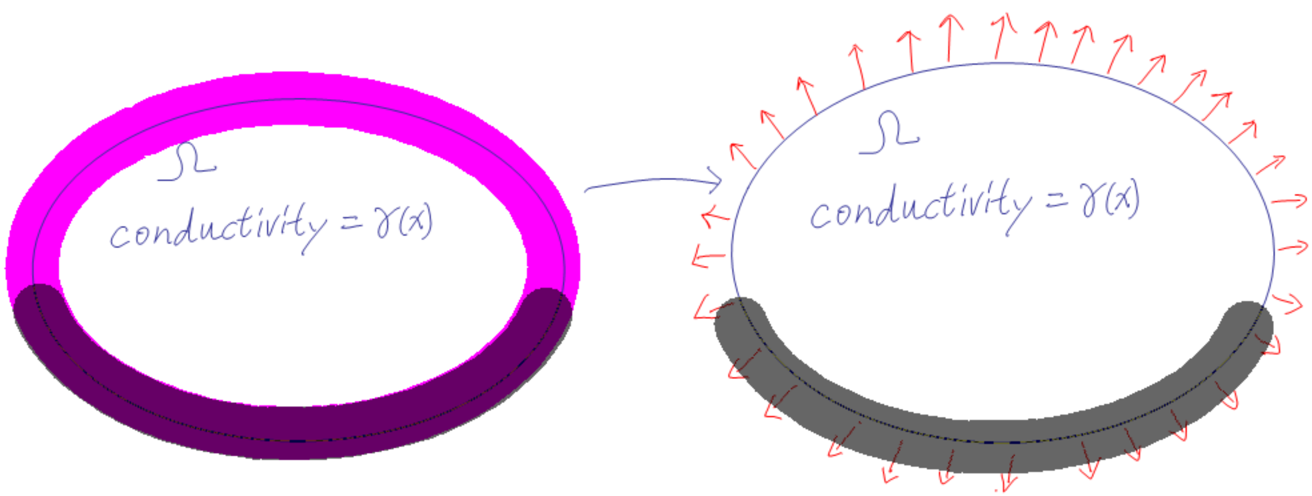
What if part of the boundary is inaccessible?



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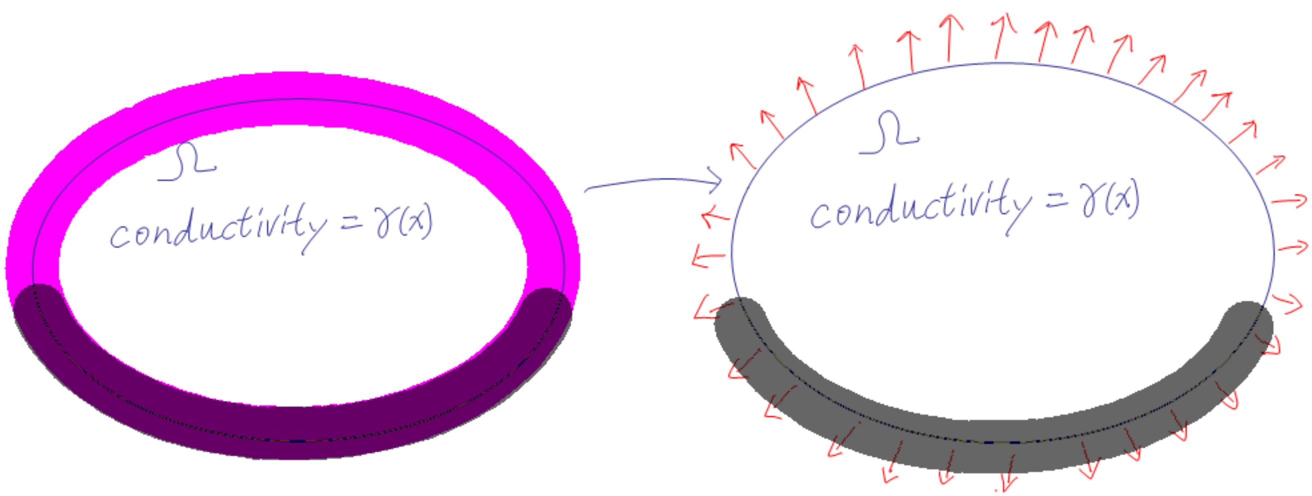

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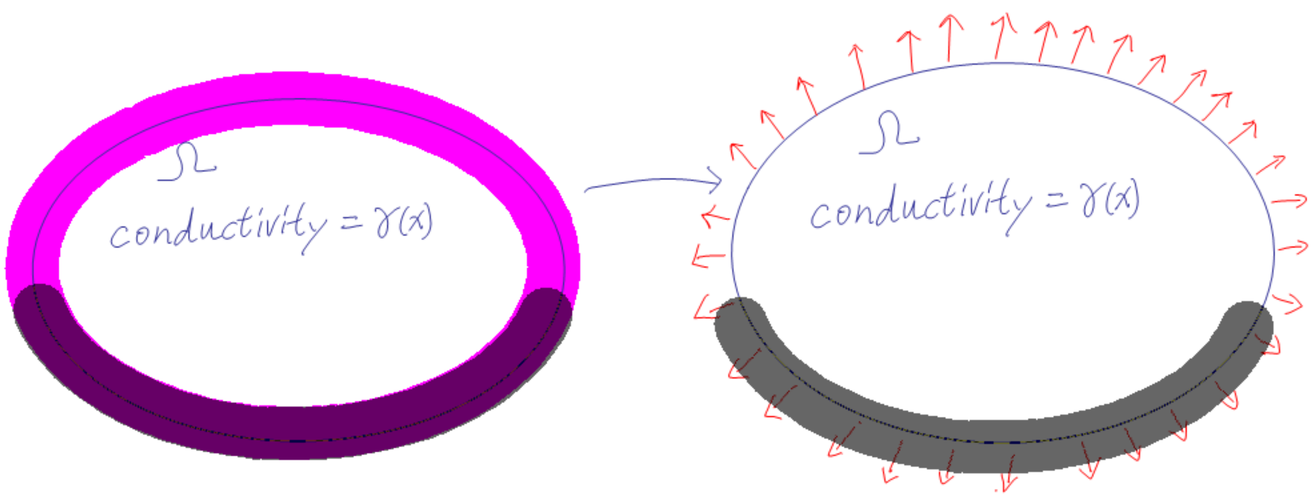

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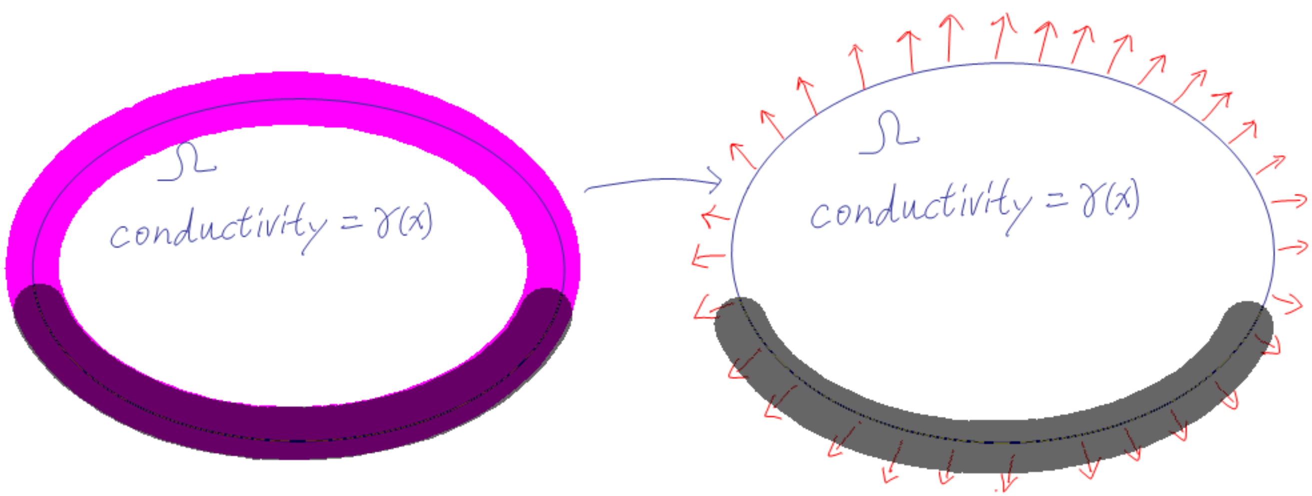

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- No explicit expression for holomorphic functions

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- No explicit expression for holomorphic functions
- Placement of critical points
- Limited data

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- Unique critical point at a given point $p \in M$.
- Φ needs to be constructed using abstract machinery

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by stationary phase.

6. $V_1(p) = V_2(p)$ at the critical point p of Φ. Move the critical point around and we have it for all points on M.

Construction of Special Solutions

We want to construct $(\Delta_g + V)u = 0$

$$u = \underbrace{exponential\ leading\ term}_{geometry} + \underbrace{remainder}_{analysis}$$

$$u \mid_{\Gamma^c} = 0$$

We first consider "free solutions" of this form when $V = 0$.

Reflected Waves (Imanuvilov-Uhlmann-Yamamoto

Suppose Φ and a are holomorphic with

$$\Phi\,|_{\Gamma^c} \in \mathbb{R} \qquad a\,|_{\Gamma^c} \in \mathbb{R}$$

then

$$\tilde{u} := \underbrace{e^{\Phi/h} a}_{incoming\ wave} - \underbrace{e^{\bar{\Phi}/h} \bar{a}}_{reflected\ wave}$$

is harmonic with

$$\tilde{u}\,|_{\Gamma^c} = 0$$

Once such a free solution is constructed, we can use Carleman estimates to solve for the remainder to get

$$u = \tilde{u} + remainder$$

$$(\Delta_g + V)u = 0$$

Conditions for Φ

So Φ has to satisfy

- $\bar{\partial}\Phi = 0$
- $\Phi \,|_{\Gamma^c} \in \mathbb{R}$
- Φ is MORSE

Recall that we can conclude $V_1(p) = V_2(p)$ ONLY IF p is the critical point of such a Φ.

So for all $p \in M$ we need such a Φ such that $\partial\Phi(p) = 0$.

(Holomorphic functions are very rigid!!)

Geometrical Point of View

We look for a section of the trivial bundle

$$E = M \times \mathbb{C}$$

- which is purely real on $\Gamma^c \subset \partial M$
- and is in the kernel of $\bar{\partial}$ operator.

So we are interested in understanding $Ker(\bar{\partial})$ in the space

$$H_F^k(M) := \{u : M \to \mathbb{C} \mid u\,|_{\partial M} \in F\}$$

where $F \subset E\,|_{\partial M}$ is a (real) rank 1 sub-bundle such that $F\,|_{\Gamma^c} = \Gamma^c \times \mathbb{R}$.

Maslov Index and $Ker(\bar{\partial})$, $Range(\bar{\partial})$

Let $E = M \times \mathbb{C}$ be the trivial bundle and

$$F \subset E\,|_{\partial M}$$

be a (real) rank 1 sub-bundle over ∂M.
The MASLOV INDEX $\mu(F, E)$ measures the winding number of F.

Let $Ker_F(\bar{\partial}) := Ker(\bar{\partial}) \cap H^k_F(M)$. Then for $\mu(F, E) + 2\chi(M) > 0$,

$$dim(Ker_F(\bar{\partial})) = \mu(F, E)$$

$$\bar{\partial} : H^k_F(M) \to holomorphic\ 1 - forms$$

is surjective.

- In our case, we require that $F\mid_{\Gamma^c} = \Gamma^c \times \mathbb{R}$.

- However, on $\Gamma \subset \partial M$ we have no requirements.

- So by letting F wind on Γ, we can make $\mu(F, E)$ as large as we wish

Therefore we have as many holomorphic functions satisfying our boundary condition as we like.

Using surjectivity, we can control the series expansion of our holomorphic function at any given point.

Consider the Map

$$\underbrace{Ker_F(\bar{\partial})}_{dim \sim \mu(F,E)} \rightarrow \underbrace{\mathbb{C}T_p^*M}_{dim=4}$$

$$u \mapsto du(p)$$

The kernel of this map is very large.

Proposition

For all $p \in M$ there exists a nontrivial holomorphic function Φ such that $\partial\Phi(p) = 0$ and $\Phi\,|_{\Gamma^c} \in \mathbb{R}$.