Gaussian heat kernel estimates : from functions to forms

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$e^{-t\Delta}$ has a smooth kernel $p_t(x,y) = p_t(y,x) > 0$:

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No curvature assumptions, rather direct geometric properties of M

Uniform bounds of the heat kernel: the polynomial case

Assume $(e^{-tL})_{t>0}$ is uniformly bounded on $L^1(M,\mu)$ ($L^\infty(M,\mu)$)

$$\sup_{x,y\in M} p_t(x,y) \leq Ct^{-D/2}, \ \forall\, t>0, \ x\in M, \text{ some } D>0,$$

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is equivalent to:
- the Sobolev inequality:

$$\|f\|_{\alpha D/(D-\alpha p)} \le C\|L^{\alpha/2}f\|_p, \quad \forall f\in \mathcal{D}_p(L^{\alpha/2}),$$

for $p>1$ and $0<\alpha p<D$ [Varopoulos 1984, C. 1990].

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- the Nash inequality:

$$\|f\|_2^{2+(4/D)} \leq C\|f\|_1^{4/D}\mathcal{E}(f), \quad \forall f \in \mathcal{F}.$$

[Carlen-Kusuoka-Stroock 1987]

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-the Gagliardo-Nirenberg type inequalities, for instance

$$\|f\|_q^2 \le C\|f\|_2^{2-\frac{q-2}{q}D}\mathcal{E}(f)^{\frac{q-2}{2q}D}, \quad \forall f\in\mathcal{F},$$

for $q>2$ such that $\frac{q-2}{2q}D<1$ [C. 1992].

Extrapolation

In the Sobolev and in the Gagliardo-Nirenberg case (not in the Nash case), one needs:

Lemma (C., 1990)

Assume $(e^{-tL})_{t>0}$ *is uniformly bounded on* $L^1(M,\mu)$ *and there exist* $1 \leq p < q \leq +\infty$, $\alpha > 0$ *such that*

$$\|e^{-tL}\|_{p\to q} \leq Ct^{-\alpha},\ \forall\, t > 0.$$

Then

$$\|e^{-tL}\|_{1\to\infty} \leq Ct^{-\beta},\ \forall\, t > 0,$$

where $\beta = \frac{\alpha}{\frac{1}{p}-\frac{1}{q}}$.

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Indeed, for instance on manifolds with non-negative Ricci curvature, $p_t(x,x) \simeq \frac{1}{V(x,\sqrt{t})}$, where $V(x,r) = \mu(B(x,r))$, and $V(x,r)$ does vary with r

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We have to assume doubling (D) :

$$V(x,2r) \leq CV(x,r),\ \forall\, x \in M,\ r > 0 \tag{1}$$

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It follows easily that there exists $\nu > 0$ such that

$$\frac{V(x,r)}{V(x,s)} \lesssim \left(\frac{r}{s}\right)^{\nu}, \quad \forall\, x \in M,\ r \geq s > 0. \qquad (VD_\nu)$$

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It is known that if M is connected, non-compact, and satisfies (1), then the following reverse doubling condition holds: there exist $0 < \nu' \leq \nu$ such that, for all $r \geq s > 0$ and $x \in M$,

$$\left(\frac{r}{s}\right)^{\nu'} \lesssim \frac{V(x,r)}{V(x,s)}.$$

Heat kernel estimates under volume doubling 1

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which implies the on-diagonal lower Gaussian estimate

$$(DLE) \qquad p_t(x,x) \geq \frac{c}{V(x,\sqrt{t})}, \forall\, x \in M,\ t > 0$$

Heat kernel estimates under volume doubling 2

Full Gaussian lower estimate

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Gradient upper estimate

$$(G) \qquad |\nabla_x p_t(x,y)| \leq \frac{C}{\sqrt{t}V(y,\sqrt{t})}, \forall\, x,y \in M,\ t > 0$$

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All this is true on manifolds with non-negative Ricci curvature

Heat kernel estimates under volume doubling 3

Theorem

$$(DUE) \Leftrightarrow (UE) \Rightarrow (DLE) \not\Rightarrow (LE) \not\Rightarrow (G)$$

$$(G) \Rightarrow (LE) \Rightarrow (DUE)$$

Davies, Grigor'yan, [Coulhon-Sikora, Proc. London Math. Soc. 2008 and Colloq. Math. 2010] [Grigory'an-Hu-Lau, CPAM, 2008], [Boutayeb, Tbilissi Math. J. 2009]

Three levels:

- (UE)
- $(UE) + (LE) = (LY) =$ parabolic Harnack
- (G)

Application: Riesz transform

Theorem

Let M be a complete non-compact Riemannian manifold satisfying (D) and (DUE). Then

$$(R_p) \qquad \| \, |\nabla f| \, \|_p \leq C \|\Delta^{1/2} f\|_p, \ \forall \, f \in \mathcal{C}_0^\infty(M),$$

for $1 < p < 2$.

[Coulhon, Duong, T.A.M.S. 1999]

Theorem

Let M be a complete non-compact Riemannian manifold satisfying (D) and (G). Then the equivalence

$$(E_p) \qquad \| \, |\nabla f| \, \|_p \simeq \|\Delta^{1/2} f\|_p, \ \forall \, f \in \mathcal{C}_0^\infty(M),$$

holds for $1 < p < \infty$.

[Auscher, Coulhon, Duong, Hofmann, Ann. Sc. E.N.S. 2004]

Pointwise heat kernel upper estimates revisited 1

Joint work with Salahaddine Boutayeb and Adam Sikora, 2013.

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$v : M \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$(D_v) \qquad v(x, 2r) \leq Cv(x, r), \forall\, r > 0,\ \mu - a.e.\, x \in M$$

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$$(D'_v) \qquad v(y, r) \leq Cv(x, r), \forall\, x, y \in M,\ r > 0,\ d(x, y) \leq r$$

v may NOT be the volume function V; in fact $v \gtrsim V$, slow decays allowed

Pointwise heat kernel upper estimates revisited 2

(DUE^v): $(e^{-t\Delta})_{t>0}$ has a measurable kernel p_t, that is

$$e^{-t\Delta}f(x) = \int_M p_t(x,y)f(y)d\mu(y),\ t>0,\ f \in L^2(M,\mu),\ \mu - a.e.\ x \in M$$

and

$$p_t(x,y) \leq \frac{C}{\sqrt{v(x,\sqrt{t})v(y,\sqrt{t})}},\ \text{for all } t>0,\ \mu - a.e.\ x,y \in M.$$

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Pointwise heat kernel upper estimates revisited 3

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Introduce

$$(N^v) \qquad \|f\|_2^2 \lesssim \|f v_r^{-1/2}\|_1^2 + r^2 \mathcal{E}(f), \quad \forall\, r > 0, \quad \forall f \in \mathcal{F}.$$

(equivalent to Nash if $v(x, r) \simeq r^D$) and

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$$(GN_q^v) \qquad \|f v_r^{\frac{1}{2}-\frac{1}{q}}\|_q^2 \lesssim \|f\|_2^2 + r^2\mathcal{E}(f), \quad \forall\, r > 0, \quad \forall f \in \mathcal{F},$$

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Theorem

Assume that (M,d,μ,L) satisfies (D) and Davies-Gaffney and that v satisfies (D_v) and (D'_v). Then (DUE^v) is equivalent to (N^v) and to (GN_q^v) for $q>2$ small enough.

Idea of the proof

Introduce weighted $L^p - L^q$ inequalities: $1 \leq p \leq q \leq +\infty$, γ, δ real numbers such that $\gamma + \delta = \frac{1}{p} - \frac{1}{q}$

$$\sup_{t>0} \| v_{\sqrt{t}}^{\gamma} \, e^{-t\Delta} \, v_{\sqrt{t}}^{\delta} \|_{p \to q} < +\infty. \qquad (vEv_{p,q,\gamma})$$

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$(DUE^v) = v_{\sqrt{t}}^{1/2}(x) p_t(x,y) v_{\sqrt{t}}^{1/2}(y) \leq C$ is equivalent to $(vEv)_{1,\infty,1/2}$ or

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Finite propagation speed of the associated wave equation$\Rightarrow$ commutation between the semigroup and the volume: for p, q fixed, equivalence between $(vEv_{p,q,\gamma})$$\Rightarrow$ extrapolation: pass from q to ∞.

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Conclusion: $(GN_q^v) \Rightarrow (DUE^v)$

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d, δ

$$\vec{\Delta} = d\delta + \delta d$$

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Bochner's formula:

$$\vec{\Delta} = \nabla^* \nabla + \mathrm{Ric}.$$

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for some $C > 0$. Here $\vec{p}_t(x,y)$ is a linear operator from T_y^*M to T_x^*M, endowed with the Riemannian metrics at y and x, and $|\cdot|$ is its norm.

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Implies (G)
Manifolds with non-negative Ricci:

$$|\vec{p}_t(x,y)| \leq p_t(x,y)$$

$$|e^{-t\vec{\Delta}}\omega| \leq e^{-t\Delta}|\omega|$$

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Joint work with Baptiste Devyver and Adam Sikora, in preparation.

A potential $\mathcal{V} \in L^\infty_{loc}$ is said to belong to the Kato class at infinity $K^\infty(M)$ if

$$\lim_{R\to\infty} \sup_{x\in M} \int_{M\setminus B(x_0,R)} G(x,y)|\mathcal{V}(y)|\, d\mu(y) = 0, \tag{2}$$

for some (all) $x_0 \in M$.

Theorem

Let M be a complete non-compact connected manifold satisfying (D) and (DUE) and such that $|\mathrm{Ric}_-| \in K^\infty(M)$. Let ν' be the reverse doubling exponent. If $\nu' > 4$, the heat kernel of $\vec{\Delta}$ satisfies $(\vec{UE})$ if and only if $\mathrm{Ker}_{L^2}(\vec{\Delta}) = \{0\}$.

Consequences

Recall the Gaussian lower bound

$$p_t(x,y) \gtrsim \frac{1}{V(x,\sqrt{t})} \exp\left(-\frac{d^2(x,y)}{ct}\right), \quad \forall\, t>0, \text{ a.e. } x,y \in M \qquad (LE)$$

Corollary

Under the above assumptions, (LE) holds.

Corollary

Under the above assumptions, (E_p) holds for all $p \in (1, +\infty)$.

Sketch of proof 1

Since $\mathrm{Ric}_- \in K^\infty(M)$, there is a compact subset K_0 of M such that

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Let R be the section of the vector bundle $\mathcal{L}(T^*M)$ given by

$$x \to R(x) = \mathrm{Ric}_-(x)\mathbf{1}_{K_0}(x).$$

We shall also denote by R the associated operator on one-forms. Set

$$H = \nabla^*\nabla + \mathrm{Ric}_+ - (\mathrm{Ric}_-)\mathbf{1}_{M\setminus K_0},$$

so that

$$\vec{\Delta} = H - R.$$

Sketch of proof 2

Lemma

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For $\lambda > 0$, we introduce the two operators

$$A_\lambda = R^{1/2} (H + \lambda)^{-1} R^{1/2}$$

and

$$B_\lambda = (H + \lambda)^{-1} R.$$

Spectral theory

Lemma

For any $\lambda \in [0,\infty)$, B_λ *is compact on* L^∞, $\sup_{\lambda\geq 0} \|B_\lambda\|_{\infty\to\infty} < \infty$, *and the map* $\lambda \mapsto B_\lambda \in \mathcal{L}(L^\infty, L^\infty)$ *is continuous on* $[0,\infty)$.

Lemma

For every $\lambda \geq 0$, *the operator* A_λ *is self-adjoint and compact on* L^2. *Furthermore,* $\mathrm{Ker}_{L^2}(\vec{\Delta}) = \{0\}$ *if and only if there is* $\eta \in (0,1)$ *such that for all* $\lambda \geq 0$,

$$\|A_\lambda\|_{2\to 2} \leq 1-\eta.$$

Lemma

Assume that $\mathrm{Ker}_{L^2}(\vec{\Delta}) = \{0\}$. *If* $\eta \in (0,1)$ *is as above then the spectral radius of* B_λ *on* L^∞ *satisfies*

$$r_\infty(B_\lambda) \leq 1-\eta, \ \forall\, \lambda \geq 0.$$

Weighted $L^p - L^q$ inequalities again

Start from

$$\sup_{t>0} \| (I + t\vec{\Delta})^{-1} V_{\sqrt{t}}^{1/p_0} \|_{p_0 \to \infty} < +\infty \qquad (RV_{p,\infty})$$

By duality and interpolation,

$$\sup_{t>0} \| V_{\sqrt{t}}^{\gamma} (I + t\vec{\Delta})^{-1} V_{\sqrt{t}}^{\delta} \|_{p \to q} < +\infty \qquad (VRV_{p,q,\gamma})$$

for any p, q such that $1 \leq p \leq p_0$, $\frac{1}{p} - \frac{1}{q} = \gamma + \delta = \frac{1}{p_0}$, $\gamma = \frac{1}{(p_0 - 1)q}$, and $\gamma + \delta = \frac{1}{p} - \frac{1}{q}$.

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Use the finite propagation speed to iterate (instead of extrapolating)