A nonlinear Brascamp–Lieb inequality

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The Brascamp–Lieb inequality

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$$\int_{\mathbb{R}^n} \prod_{j=1}^{m} f_j(L_j x)^{c_j}\, \mathrm{d}x \le C \prod_{j=1}^{m} \left(\int_{\mathbb{R}^{n_j}} f_j \right)^{c_j}$$

$$f_j \in L^1(\mathbb{R}^{n_j}),\ f_j \ge 0$$

The Brascamp–Lieb inequality

$$\int_{\mathbb{R}^n} \prod_{j=1}^m f_j(L_j x)^{c_j}\,\mathrm{d}x \le \mathrm{B}(\mathbf{L}) \prod_{j=1}^m \left(\int_{\mathbb{R}^{n_j}} f_j\right)^{c_j}$$

$f_j \in L^1(\mathbb{R}^{n_j}),\ f_j \ge 0$

$L_j : \mathbb{R}^n \to \mathbb{R}^{n_j}$ linear

$c_j \in (0,1]$

$(\mathbf{L},\mathbf{c})$: Brascamp–Lieb data $\mathbf{L} = (L_j)_{j=1}^m,\ \mathbf{c} = (c_j)_{j=1}^m$

$\mathrm{B}(\mathbf{L}) \in [0,\infty]$: Brascamp–Lieb constant best constant

$$\mathrm{B}(\mathbf{L}) = \sup_{\int f_j = 1} \int_{\mathbb{R}^n} \prod_{j=1}^m f_j(L_j x)^{c_j}\,\mathrm{d}x$$

$\mathrm{B}(\mathbf{L}) < \infty \Rightarrow n = \sum_{j=1}^m c_j n_j$ and each L_j is surjective

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Characterisation of finiteness

Theorem (Bennett–Carbery–Christ–Tao)

$\mathrm{B}(\mathbf{L}) < \infty$ if and only if

(i) $n = \sum_{j=1}^{m} c_j n_j$

(ii) $\dim(V) \leq \sum_{j=1}^{m} c_j \dim(L_j V)$ for all $V \leq \mathbb{R}^n$

Special role of gaussians

Theorem (Lieb)

$$\mathrm{B}(\mathbf{L}) = \sup_{A_j > 0} \frac{\prod_{j=1}^m \det(A_j)^{c_j/2}}{\det(\sum_{j=1}^m c_j L_j^* A_j L_j)^{1/2}}$$

Note

$$\int_{\mathbb{R}^n} \prod_{j=1}^m f_j(L_j x)^{c_j}\, \mathrm{d}x = \frac{\prod_{j=1}^m \det(A_j)^{c_j/2}}{\det(\sum_{j=1}^m c_j L_j^* A_j L_j)^{1/2}}$$

where

$$f_j(x) = (\det A_j)^{\frac{1}{2}} \exp(-\pi \langle A_j x, x\rangle)$$

The Loomis–Whitney inequality

$$\int_{\mathbb{R}^n} \prod_{j=1}^{n} f_j(\Pi_j x)^{\frac{1}{n-1}} \,\mathrm{d}x \le \prod_{j=1}^{n} \left(\int_{\mathbb{R}^{n-1}} f_j \right)^{\frac{1}{n-1}}$$

Here $\Pi_j x = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n)$

- $\ker \Pi_j = \mathrm{span}(e_j)$
- If $\ker \widetilde{\Pi}_j = \mathrm{span}(v_j)$ and $\mathrm{span}(v_1, \dots, v_n) = \mathbb{R}^n$ then

$$\int_{\mathbb{R}^n} \prod_{j=1}^{n} f_j(\widetilde{\Pi}_j x)^{\frac{1}{n-1}} \,\mathrm{d}x \le C \prod_{j=1}^{n} \left(\int_{\mathbb{R}^{n-1}} f_j \right)^{\frac{1}{n-1}}$$

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Stability of the Brascamp–Lieb constant

Recall

$$\mathrm{B}(\mathbf{L}) = \sup_{\int f_j = 1} \int_{\mathbb{R}^n} \prod_{j=1}^{m} f_j(L_j x)^{c_j}\, \mathrm{d}x$$

Theorem (Bennett–B–Cowling–Flock)

$\mathbf{L} \mapsto \mathrm{B}(\mathbf{L})$ is continuous

The nonlinear Brascamp–Lieb inequality

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$$\int_U \prod_{j=1}^m f_j(\varphi_j(x))^{c_j}\,\mathrm{d}x \leq C \prod_{j=1}^m \left(\int_{\mathbb{R}^{n_j}} f_j\right)^{c_j} \qquad \text{(NBL)}$$

$f_j \in L^1(\mathbb{R}^{n_j}),\ f_j \geq 0$

$\varphi_j : \mathbb{R}^n \to \mathbb{R}^{n_j}$ C^2 submersion near 0

$U \subseteq \mathbb{R}^n$

Conjecture (local version)

If $\mathrm{B}(\mathbf{L^0}) < \infty$ where $L_j^0 = \mathrm{d}\varphi_j(0)$, then there exists a neighbourhood $U \ni 0$ and $C < \infty$ such that (NBL) holds

Nonlinear Loomis–Whitney inequality (Bennett–Carbery–Wright)

Conjecture holds with $L_j^0 = \Pi_j$

- B–C–W proof : Christ's method of refinements + tensorisation
- Further proofs : Bejenaru–Herr–Tataru (induction-on-scales), Koch–Steinerberger, Carbery–Hänninen–Valdimarsson (under C^1 regularity, holds with $C = 1 + \varepsilon$ on a neighbourhood U_ε)
- Nonlinear LW yields multilinear singular convolution estimates and these were applied to wellposedness of Zakharov system on $\mathbb{R}^2 \times \mathbb{R}$

$$i\partial_t u + \Delta u = vu$$
$$\Box v = \Delta |u|^2$$

  by Bejenaru–Herr–Holmer–Tataru
- Further applications of nonlinear LW (type) : Bejenaru–Herr, Kinoshita, Hirayama–Kinoshita, Kinoshita–Schippa,...

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Induction-on-scales argument

Define $\Lambda(\mathbf{f}) = \int_{\mathbb{R}^n} \prod_{j=1}^m (f_j \circ L_j)^{c_j} \div \prod_{j=1}^m \left(\int f_j \right)^{c_j}$

Suppose $\int f_j = \int g_j = 1$, and setting $h_j^x(z) = f_j(z) g_j(L_j x - z)$,

$$\Lambda(\mathbf{f})\Lambda(\mathbf{g}) = \int \prod_j f_j(L_j y)^{c_j} \int \prod_j g_j(L_j(x-y))^{c_j} \, \mathrm{d}x \, \mathrm{d}y$$

$$= \int \left(\int \prod_j (h_j^x(L_j y))^{c_j} \, \mathrm{d}y \right) \mathrm{d}x$$

$$= \int \Lambda(\mathbf{h}^x) \prod_j (f_j * g_j(L_j x))^{c_j} \, \mathrm{d}x$$

$$\leq \sup_x \Lambda(\mathbf{h}^x) \int \prod_j (f_j * g_j(L_j x))^{c_j} \, \mathrm{d}x$$

$$= \sup_x \Lambda(\mathbf{h}^x) \Lambda(\mathbf{f} * \mathbf{g})$$

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&= \int \left(\int \prod_j (h_j^x(L_j y))^{c_j} \, \mathrm{d}y \right) \mathrm{d}x \\
&= \int \Lambda(\mathbf{h}^x) \prod_j (f_j * g_j(L_j x))^{c_j} \, \mathrm{d}x \\
&\leq \sup_x \Lambda(\mathbf{h}^x) \int \prod_j (f_j * g_j(L_j x))^{c_j} \, \mathrm{d}x \\
&= \sup_x \Lambda(\mathbf{h}^x) \Lambda(\mathbf{f} * \mathbf{g})
\end{aligned}$$

Ball's inequality

If $h_j^x(z) = f_j(z) g_j(L_j x - z)$ then

$$\Lambda(\mathbf{f}) \leq \frac{\sup_x \Lambda(\mathbf{h}^x) \Lambda(\mathbf{f} * \mathbf{g})}{\Lambda(\mathbf{g})}$$

- If we additionally assume that $\mathbf{g}$ is a **maximiser** then

$$\Lambda(\mathbf{f}) \leq \sup_x \Lambda(\mathbf{h}^x)$$

  - $\mathbf{h}^x$ is a certain "localised" version $\mathbf{f}$ (w.r.t. maximiser $\mathbf{g}$)
  - If g_j has compact (tiny) support near 0, then $h_j^x \approx f_j$ near $L_j x$
  - Strong indication we should try to induct on size of $\operatorname{supp} \mathbf{f}$

- Or, $\Lambda(\mathbf{f}) \leq \Lambda(\mathbf{f} * \mathbf{g}) \quad \rightsquigarrow$ induct on scale of constancy of $\mathbf{f}$

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Define $\mathcal{C}(\delta) = \sup_{\int f_j = 1} \int_{B(0,\delta)} \prod_{j=1}^{m} (f_j \circ \varphi_j)^{c_j}$

Optimistic hope: Something like...

$\mathcal{C}(\delta) \leq (1+\delta^{\beta})\mathcal{C}(\delta^{\alpha})$ (some $\alpha \in (1,2)$, $\beta > 0$)

Recall $L_j^0 = \mathrm{d}\varphi_j(0)$, and let's normalise $\varphi_j(0) = 0$, $\int f_j = 1$

As in the proof of Ball's inequality ($F = \prod_{j=1}^{m} (f_j \circ \varphi_j)^{c_j}$)

$$\mathrm{B}(\mathbf{L}^0) \int_{B(0,\delta)} F(y)\,\mathrm{d}y = \int_{B(0,\delta)} F(y)\,\mathrm{d}y \int_{\mathbb{R}^n} \prod_j g_{\delta,j}(L_j^0 x)^{c_j}\,\mathrm{d}x$$

if g is a gaussian maximiser for $\mathbf{L}^0$, and

$$g_{\delta,j}(w) = \delta^{-\alpha' n_j} g_j(\delta^{-\alpha'} w) \qquad (\alpha' > \alpha, \int g_j = 1)$$

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$$\leq (1+\delta^\beta)\int_{B(0,\delta)} F(y)\,\mathrm{d}y \int_{B(0,\delta^\alpha)}\prod_j g_{\delta,j}(L_j^0 x)^{c_j}\,\mathrm{d}x$$

$$= (1+\delta^\beta)\int_{B(0,\delta)} F(y)\int_{B(y,\delta^\alpha)}\prod_j g_{\delta,j}(L_j^0(x-y))^{c_j}\,\mathrm{d}x\mathrm{d}y$$

Now $L_j^0(x-y) = L_j^0 x - \varphi_j(y) + O(\delta^2)$ so

$$\cdots \leq (1+\delta^\beta)^2\int_{B(0,2\delta)}\int_{B(x,\delta^\alpha)}\prod_j h_j^x(\varphi_j(y))^{c_j}\,\mathrm{d}y\mathrm{d}x$$

where $h_j^x(z) = f_j(z)g_{\delta,j}(L_j^0 x - z)$

So, with $h_j^x(z) = f_j(z) g_{\delta,j}(L_j^0 x - z)$,

$$\begin{aligned}
&\mathrm{B}(\mathbf{L}^0) \int_{B(0,\delta)} \prod_{j=1}^m f_j(\varphi_j(y))^{c_j}\, \mathrm{d}y \\
&\leq (1+\delta^\beta) \int_{B(0,2\delta)} \int_{B(x,\delta^\alpha)} \prod_j h_j^x(\varphi_j(y))^{c_j}\, \mathrm{d}y \mathrm{d}x \\
&\leq (1+\delta^\beta) \int_{B(0,2\delta)} \mathcal{C}(x,\delta^\alpha) \prod_j \left(\int h_j^x \right)^{c_j} \mathrm{d}x
\end{aligned}$$

where

$$\mathcal{C}(u,\delta) = \sup_{\int f_j = 1} \int_{B(u,\delta)} \prod_{j=1}^m (f_j \circ \varphi_j)^{c_j}$$

Note $\int h_j^x = f_j * g_{\delta,j}(L_j^0 x)$, so

$$\int \prod_j \left(\int h_j^x \right)^{c_j} \mathrm{d}x \leq \mathrm{B}(\mathbf{L}^0)$$

An argument like the above gives

$$\int_{B(u,\delta)} \prod_{j=1}^{m} (f_j \circ \varphi_j)^{c_j} \leq (1+\delta^\beta) \sup_{x \in B(u,2\delta)} \mathcal{C}(x, \delta^\alpha)$$

and thus

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Big issues to deal with

- At the very start we assumed gaussian maximisers exist – not always the case!
- Lieb's theorem guarantees gaussian near-maximisers but to keep the argument tight, we need a quantitative version of Lieb's theorem

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$$\mathcal{C}(u,\delta) \le (1+\delta^{\beta}) \max_{x \in B(u,2\delta)} \mathcal{C}(x,\delta^{\alpha})$$

want to zoom in enough to do something like

$$f_j(\varphi_j(x)) \le \kappa f_j(\mathrm{d}\varphi_j(u)x) \qquad (x \in B(u,\delta))$$

Recall $\mathcal{C}(u,\delta) = \sup_{\int f_j = 1} \int_{B(u,\delta)} \prod_{j=1}^{m} (f_j \circ \varphi_j)^{c_j}$

- For this, need f_j "locally constant" so ... need more parameters (functions κ-constant at scale μ) and keep track of how these evolve during the induction
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Theorem (Bennett–B–Buschenhenke–Cowling–Flock)

Suppose $\varphi_j : \mathbb{R} \to \mathbb{R}^{n_j}$ is a C^2 submersion near 0 s.t. $\mathrm{B}(\mathbf{L^0}) < \infty$, where $L_j^0 = \mathrm{d}\varphi_j(0)$.

Then $\forall \varepsilon > 0$, $\exists U \ni 0$ s.t.

$$\int_U \prod_{j=1}^m (f_j \circ \varphi_j)^{c_j} \leq (1+\varepsilon)\mathrm{B}(\mathbf{L^0}) \prod_{j=1}^m \left(\int_{\mathbb{R}^{n_j}} f_j \right)^{c_j}$$