Large N Limit of the $O(N)$ Linear Sigma Model via Stochastic Quantization

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Joint work with Hao Shen, Scott Smith and Xiangchan Zhu

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Gaussian free field

- Recall the free field in the Euclidean quantum field theory. The usual free field on the torus $\mathbb{T}^d$ is heuristically described by the following probability measure:

$$\nu(\mathrm{d}\Phi) = C_N^{-1}\Pi_{x\in\mathbb{T}^d}\mathrm{d}\Phi(x)\exp\Big(-\int_{\mathbb{T}^d}(|\nabla\Phi|^2 + m\Phi^2)\mathrm{d}x\Big),$$

where C_N is the normalization constant and Φ is the real-valued field.

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- This corresponds to the Gaussian measure $\nu := \mathcal{N}(0,(m-\Delta)^{-1})$ rigorously defined on $\mathcal{S}'$.
- The free field describes particles which do not interact.

Φ^4_d field

The Φ^4_d model is the simplest non-trivial Euclidean quantum field:

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- Other models

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Stochastic quantization on $\mathbb{T}^d$, $d = 2, 3$:

$$\mathcal{L}\Phi_i = -\frac{1}{N}\sum_{j=1}^N \Phi_j^2 \Phi_i + \xi_i,$$

$\mathcal{L} = \partial_t - \Delta + m$; $(\xi_i)_{i=1}^N$: independent space-time white noises.

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Questions: Large N limit of the dynamics Φ_i and the field ν^N?

Stochastic quantization: Da Prato-Debussche trick

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- $\Phi_i \in C^-$ for $d=2$; $\Phi_i \in C^{-\frac{1}{2}-}$ for $d=3$
- Decompose $\Phi_i = Y_i + Z_i$ as Da Prato-Debussche trick for $d=2$

$$\mathcal{L}Z_i = \xi_i,$$

$$\mathcal{L}Y_i = -\frac{1}{N}\sum_{j=1}^N (Y_j^2 Y_i + Y_j^2 Z_i + 2Y_j Y_i Z_j + 2Y_j \underbrace{:Z_iZ_j:}_{\text{Wick product}} + \underbrace{:Z_j^2:}_{\text{Wick product}} Y_i + \underbrace{:Z_iZ_j^2:}_{\text{Wick product}}),$$

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- $Z_i \in C^-$, $Y_i \in C^{2-}$; Wick product: $:Z_iZ_j: = Z_iZ_j - \mathbf{E}Z_iZ_j$.

Difficulty for $d = 3$

- Decompose $\Phi_i = Z_i + Y_i$ as $d = 2$

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- Key point: The red terms are not well defined even we do further decomposition!

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- Key point: The red terms are not well defined even we do further decomposition!
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- This is not enough since the stopping time may depend on N

Limiting equation and convergence of the dynamics when $d = 2$

- The dynamical linear sigma model

$$\mathcal{L}\Phi_i = -\frac{1}{N}\sum_{j=1}^{N}(\Phi_j^2 - \mathbf{E}[Z_i^2])\Phi_i + \xi_i, \quad \Phi_i(0) = \phi_i$$

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$$\mathcal{L}\Psi_i = -\mu\Psi_i + \xi_i, \quad \Psi_i(0) = \psi_i,$$

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Theorem [Shen, Scott, Zhu, Z 20]

Suppose that $d = 2$ and (ψ_i, ψ_j) are independent and have the same law and for $p > 1$ $\mathbf{E}\|\phi_i - \psi_i\|^p_{C^{-\kappa}} \to 0$, as $N \to \infty$.

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Suppose that $d = 2$ and (ψ_i, ψ_j) are independent and have the same law and for $p > 1$ $\mathbf{E}\|\phi_i - \psi_i\|^p_{C^{-\kappa}} \to 0$, as $N \to \infty$. It holds that for $t > 0$, $\mathbf{E}\|\Phi_i(t) - \Psi_i(t)\|^2_{L^2} \to 0$ and $\|\Phi_i - \Psi_i\|_{C_T C^{-1}} \to^P 0$, as $N \to \infty$.

- Mean field limit/ Propagation of chaos

Idea of Proof: Uniform bounds

$\Phi_i = Z_i + Y_i$, $\Psi_i = Z_i + X_i$

$$\mathcal{L}Z_i = \xi_i,$$

$$\mathcal{L}Y_i = -\frac{1}{N}\sum_{j=1}^{N}(Y_j^2 Y_i + Y_j^2 Z_i + 2Y_jZ_jY_i + 2Y_j :Z_jZ_i: + :Z_j^2: Y_i + :Z_iZ_j^2:),$$

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Lemma 1

It holds that for $p \geq 2$

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Invariant measure to Limiting equation

- The limiting equation

$$(\partial_t - \Delta + m)\Psi_i = \mathcal{L}\Psi_i = -\mu\Psi_i + \xi_i,$$

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For $d = 1, 2$, there exists $m_0 > 0$ such that: for $m \geq m_0$, the Gaussian free field $\mathcal{N}(0, (m - \Delta)^{-1})$ is the unique invariant measure to Ψ.

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Convergence of invariant measure (field)

- $O(N)$ linear sigma model:

$$\nu^N = \frac{1}{C_N}\exp\Big(-2\int_{\mathbb{T}^d}\frac{1}{2}\sum_{j=1}^N|\nabla\Phi_j|^2+\frac{m}{2}\sum_{j=1}^N\Phi_j^2+\frac{1}{4N}\Big(\sum_{j=1}^N\Phi_j^2\Big)^2\mathrm{d}x\Big)\mathcal{D}\Phi,$$

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For $d = 2, 3$

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Idea of proof: $\mathbb{W}_2(\nu^{N,i},\nu) \lesssim N^{-\frac{1}{2}}$ for $d=3$

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- Cancelation from paraprodcts [Gubinelli, Hofmanova 18]

$$\sum_{i=1}^{N}\Big[\langle(\Delta - m)Y_i, Y_i\rangle - \frac{1}{N}\sum_{j=1}^{N}\langle 2Y_j \preccurlyeq : Z_iZ_j : + Y_i \preccurlyeq : Z_j^2 :, Y_i\rangle\Big]$$

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$$\leq\mathbf{E}\Big(\sum_{j=1}^N\|Y_j(0)\|_{L^2}^2\Big)+C\mathbf{E}\int_0^T\Big(\sum_{i=1}^N\|Y_i\|_{L^2}^2\Big)R_N\mathrm{d}s+C.$$

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$R_N=R_N-\mathbf{E}[R_N]+\mathbf{E}[R_N]$

Observables

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Theorem [Shen, Scott, Zhu, Z. 20/ Shen, Zhu, Z. 21]

Suppose that $\Phi \simeq \nu^N$. For $\kappa > 0$, m large enough, the following result holds:

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$$\lim_{N\to\infty} \frac{1}{\sqrt{N}} \sum_{i=1}^N : \Phi_i^2 : \neq \lim_{N\to\infty} \frac{1}{\sqrt{N}} \sum_{i=1}^N : Z_i^2 :$$

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- Idea: Improved moment estimate for stationary case by independence

$$\mathbf{E}\Big[\Big(\sum_{i=1}^N \|Y_i\|_{L^2}^2\Big)^q\Big] + \mathbf{E}\Big[\Big(\sum_{i=1}^N \|Y_i\|_{L^2}^2 + 1\Big)^q \Big(\sum_{i=1}^N \|\nabla Y_i\|_{L^2}^2\Big)\Big] \lesssim 1.$$

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- $\frac{1}{N}:(\sum_{i=1}^N \Phi_i^2)^2:$ is tight in $B_{1,1}^{-3\kappa}$ for $d=2$
- For $d=1,2$,

$$\lim_{N\to\infty}\frac{1}{\sqrt{N}}\sum_{i=1}^N :\Phi_i^2: \neq \lim_{N\to\infty}\frac{1}{\sqrt{N}}\sum_{i=1}^N :Z_i^2:$$

$$\lim_{N\to\infty}\frac{1}{N}:(\sum_{i=1}^N \Phi_i^2)^2: \neq \lim_{N\to\infty}\frac{1}{N}:(\sum_{i=1}^N Z_i^2)^2:$$

- Idea: Improved moment estimate for stationary case by independence

$$\mathbf{E}\Big[\Big(\sum_{i=1}^N \|Y_i\|_{L^2}^2\Big)^q\Big] + \mathbf{E}\Big[\Big(\sum_{i=1}^N \|Y_i\|_{L^2}^2 + 1\Big)^q\Big(\sum_{i=1}^N \|\nabla Y_i\|_{L^2}^2\Big)\Big] \lesssim 1.$$

- Integration by parts formula/ Dyson-Schwinger from [Kupiainen 80]

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- Other models

Thank you !