Coupled geometric flows with harmonic map flow

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Harmonic maps

$(M, g), (N, h)$: Riemannian manifolds.
The Dirichlet energy $E(f)$ of a map $f : (M, g) \to (N, h)$ is defined by

$$E(f) = \int_M e(f) = \frac{1}{2} \int_M |df|^2 dvol_g,$$

where $e(f) = \frac{1}{2}|df|^2 dvol_g = \frac{1}{2} g^{ij} h_{\alpha\beta} f_i^\alpha f_j^\beta dvol_g$ is the energy density and $dvol_g$ is the volume form on (M, g).
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Embed $(N, h) \hookrightarrow \mathbb{R}^L$ isometrically, then f : harmonic $\Rightarrow$

$$\tau(f)(x) = \Delta_g f(x) + A_g(f(x))(df, df) = 0,$$

where $\tau(f) = \mathrm{tr}_g \nabla df \in \Gamma(f^* TX)$ is the tension field of f and A_g is the second fundamental form of the embedding.

Properties of harmonic maps

Theorem 1 (Known results of harmonic maps)

Let $f \in W^{1,2}(M, N)$ be a map with $\dim M = m$.

1. *The Dirichlet energy $E(f)$ is conformally invariant if $m = 2$.*
2. *If $m = 2$ and f is harmonic, then using local isothermal coordinate $z = x + iy$, the Hopf differential $\Phi = (|f_x|^2 - |f_y|^2 + 2i\langle f_x, f_y\rangle)dz^2$ is holomorphic.*
3. *Any harmonic map $f \in W^{1,2}(M, N)$ with $m = 2$ is smooth. (Helein, '91)*
4. *There is a harmonic map $f : B^3 \to S^2$ which is discontinuous everywhere. (Riviere, '95)*
5. *The Hausdorff dimension of singular set of harmonic map is at most $m - 2$. (Schoen, '84)*
6. *The Hausdorff dimension of singular set of minimizing harmonic map is at most $m - 3$. (Schoen-Uhlenbeck, '82)*

Heat flow of harmonic maps

Heat flow of harmonic map is the gradient flow of Dirichlet energy:

$$f_t = \tau(f) = \Delta_g f + A(f)(df, df) \tag{1}$$

with initial condition $f(0) = f_0$.

Theorem 2 (Eells-Sampson, '64)

For any $f_0 \in C^\infty(M, N)$, there is $T_0 > 0$ such that the heat flow equation (1) admits a unique, smooth solution $f \in C^\infty(M \times [0, T_0), N)$.

With additional curvature assumption on the target, we get more.

Theorem 3 (Eells-Sampson, '64)

If moreover sectional curvature of N is non-positive, then the solution exists on $M \times [0, \infty)$.

Bubbles

Without non-positive curvature assumption, we get global weak solution.

Theorem 4 (Struwe, '85)

If $\dim M = 2$, *then for any* $f_0 \in W^{1,2}(M,N)$, $\exists f : M \times [0,\infty) \to N$, *smooth except finitely many* (x_i, t_i). *Moreover, at singular point* (x,T),

$$\lim_{r \searrow 0} \lim_{t \nearrow T} E(f(t), D_r(x)) \neq 0.$$

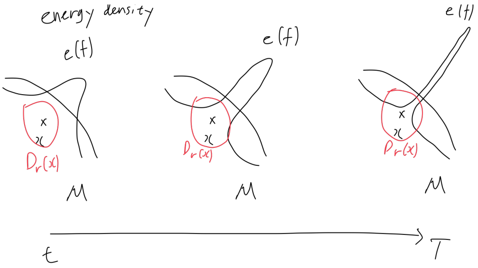
Those singular points are also called **bubble points**.
The loss of energy is captured by bubbles : $\phi_i : S^2 \to N$ such that

$$\lim_{t \nearrow T} E(f(t)) = E(f(T)) + \sum_i E(\phi_i).$$

Theorem 5 (Struwe, '85)

If $E(f_0) < \varepsilon_0$, *then the solution is smooth globally.*

Picture of bubbles



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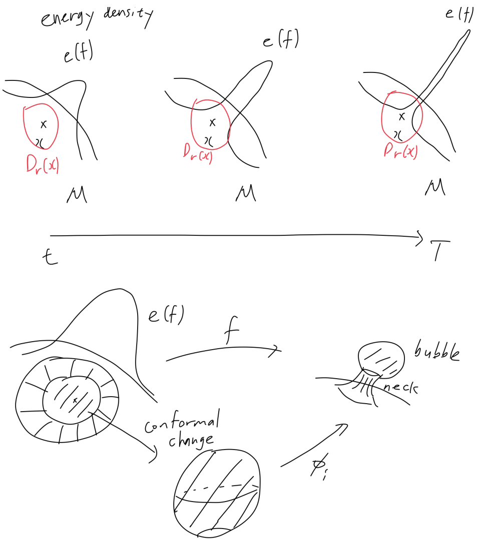

Finite time singularity

Finite time singularities Do exist!!

Theorem 6 (Chang-Ding-Ye, '92)

There exists $f_0 : D^2 \to S^2$ *such that the solution of heat flow equation* (1) *with initial map* f_0 *blows up at the origin in finite time.*

More generally,

Theorem 7 (Davila-del Pino-Wei, '20)

There exists a solution $f : \Omega \times (0, T) \to S^2$ *of heat flow equation* (1) *that blows up at* $q_1, \ldots, q_k$ *in* T.

Those bubbling points should be apart from each other.

Theorem 8 (Qing-Tian, '97)

In heat flow solution, bubbles are decoupled each other.

Non-uniqueness

Theorem 9 (Freire, '95)

If furthermore $E(f(s)) < E(f(t))$ for almost every $s < t$, then the weak solution is unique.

Remark : Energy decreasing in Struwe's sense is $\frac{d}{dt}E(f(t)) \leq 0$ for almost every t. So, Struwe's solution allows bubbling off but it also allows "reverse bubbling".

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Theorem 10 (Topping, '02)

There exists a harmonic map $f_0 : D \to S^2$ and a weak solution $f \in W^{1,2}(D \times [0,\infty), S^2)$ of the heat flow equation (1) such that $f(t) = f_0$ for all $t \in [0,1]$ but $f(t) \neq f_0$ for $t > 1$.
Moreover,

$$\lim_{t \searrow 1} E(f(t)) = E(f_0) + 4\pi.$$

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Decomposition of space of metrics, 2-dimension

Fix $g_0 \in \mathcal{M}_c$: a metric on M with constant curvature.
$\{g : \text{ smooth metric on } M\} = Sym^2_+(T^*M)$.
Its tangent space (at g_0) is

$$Sym^2(T^*M) = C(g_0) \oplus \{L_X g_0\} \oplus \mathcal{H}(g_0)$$

where

$$C(g_0) = \{\phi \cdot g_0 : \phi \in C^\infty(M)\}$$
$$\{L_X g_0\} = \{L_X g_0 : X \in \Gamma(TM)\}$$

and $\mathcal{H}(g_0)$ consists of the real parts of holomorphic quadratic differentials.

That means, change of metric can be split into change in conformal direction, Lie-derivative direction, and so called horizontal direction.

Teichmüller flow of harmonic maps

In $\dim M = 2$, conformal change of metric does not change the energy.

Lemma 11 (Rupflin-Topping, '16)

$\forall g(t) \in \mathcal{M}_c$ smooth, there is diffeomorphism $f_t : M \to M$ such that

$$\partial_t g_0(t) = \mathcal{H}(g_0(t))$$

where $g_0(t) = f_t^* g(t)$.

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Definition 1 (Rupflin-Topping, '16)

Teichmüller flow is a pair equations

$$\begin{cases} f_t & = \Delta_g f + A(f)(df, df) \\ g_t & = \frac{1}{4} P_g \Phi(f, g) \end{cases} \tag{2}$$

$\Phi(f,g)$: Hopf differential, $P_g : Sym^2(T^*M) \to \mathcal{H}(g)$: L^2 orthogonal projection.

Another type of singularity

This equation is obtained by L^2 gradient flow of

$$E(f,g) = \frac{1}{2}\int_M |df|_g^2 dvol_g.$$

So the flow decreases the energy in the fastest direction.
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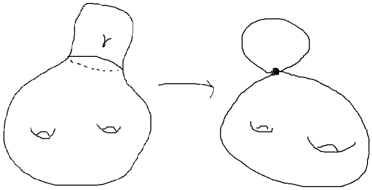

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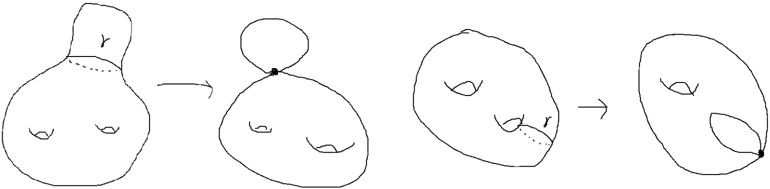

Properties of Teichmüller flow

Theorem 12

1. *A weak solution of (2) exists on $[0,T)$ for $T \in (0,\infty]$. Moreover, if $T < \infty$, then lengths of closed geodesic $\ell(g(t)) \to 0$. (Rupflin, '14)*
2. *If $\ell(g(t)) \to 0$ as $t \to \infty$, $(M, g(t))$ splits into finitely many lower genus surfaces and $f(t)$ subconverges to branched minimal immersions. (Rupflin-Topping-Zhu, '13)*
3. *In the above, the energy is not lost. (Huxol-Rupflin-Topping, '16)*
4. *If (N, h) has non-positive curvature, then there is a global smooth solution. (Rupflin-Topping, '18)*
5. *If $\ell(g(t)) \to 0$ as $t \to T$, Parts 2 and 3 hold. (Rupflin-Topping, '19)*

Remark 1

Unlike harmonic map heat flow, there can be necks connecting bubbles and body maps. (Rupflin-Topping, '19)

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Brief idea

Idea : Consider variation of metrics that does not change domain geometry. $\Rightarrow$ conformal direction!
Let $g(x,t) = e^{2u(x,t)} g_0(x)$, time-dependent metric on M.
Then over the region where $|df|^2$ becomes large, make $u(x,t)$ large.

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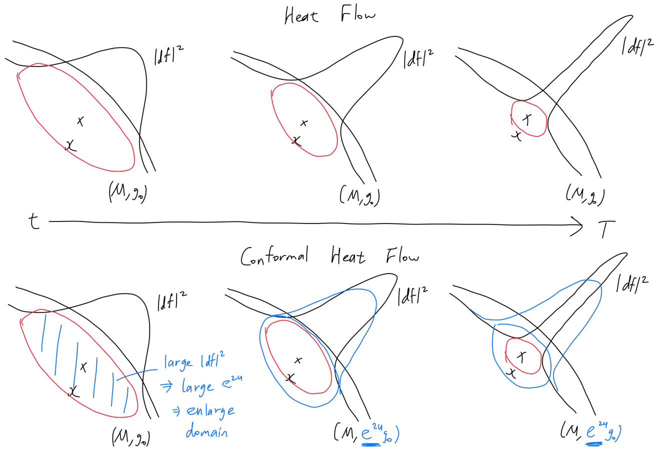

Conformal heat flow of harmonic maps

Let $f_0 : M \to N$ be a map and $g(x,t)$ be a time-dependent metric on M. For $a, b > 0$ constants, consider a pair of equations

$$\begin{cases} f_t &= \Delta_{g(t)}(f) + A_{g(t)}(f)(df, df) \\ g_t &= (2b|df|^2_{g(t)} - 2a)g \end{cases} \tag{3}$$

with initial conditions $f(0) = f_0$ and $g(0) = g_0$.
If we let $g(x,t) = e^{2u(x,t)}g_0$, the equation for f and u becomes

$$\begin{cases} f_t &= e^{-2u}(\Delta(f) + A(f)(df, df)) \\ u_t &= be^{-2u}|df|^2 - a \end{cases} \tag{4}$$

with $f(0) = f_0$ and $u(0) = 0$.
The volume is defined by

$$V(t) = \int_M dvol_{g(t)} = \int_M e^{nu} dvol_{g_0}. \tag{5}$$

Volume

Let $\dim M = 2$.

Lemma 1

For smooth solution of CHF, energy is decreasing. Volume satisfies

$$V(t) = e^{-2at}\left(V(0) + 4b\int_0^t e^{2as}E(s)ds\right) \leq e^{-2at}V(0) + \frac{2b}{a}E_0.$$

This lemma comes from direct solution of u:

$$e^{2u} = e^{-2at}\left(1 + 2b\int_0^t e^{2as}|df|^2(s)ds\right).$$

Lemma 2

If f_0 is harmonic, then $f(t) = f_0$ and the energy density becomes constant $\frac{a}{b}$ as $t \to \infty$.

Properties of conformal heat flow

Theorem 13

- *For any* $f_0 \in W^{3,2}(M,N)$, *there exist* $T_0 > 0$ *and a pair of smooth solutions* $f : M \times [0,T_0] \to N$ *and* $u : M \times [0,T_0] \to \mathbb{R}$ *of* (4).
- *A global weak solution* (f,u) *exists on* $M \times [0,\infty)$ *which is smooth on* $M \times (0,\infty)$ *except at most finitely many points* (x_i, t_i).
- *At singularity* (x,T), *there is* $(x_k,t_k) \in M \times [0,T)$ *with* $(x_k,t_k) \to (x,T)$ *such that* $|df|^2(x_k,t_k) \to \infty$.

Many questions are unanswered, like:

Question 1

1. *Does CHF avoid finite time singularity?*
2. *Is CHF unique?*
3. *Is there infinite time singularity?*

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Harmonic-Ricci flow

Let $f_0 : M \to N$ be a map and $g(x,t)$ be a time-dependent metric on M. Consider a pair of equations

$$\begin{cases} g_t &= -2Ric + 2\alpha(t) df \otimes df \\ f_t &= \tau_g(f) \end{cases} \tag{6}$$

with initial conditions $f(0) = f_0$ and $g(0) = g_0$ and α is a positive coupling time-dependent function.

An example of harmonic-Ricci flow arises in warped product manifold. Let $M = M_1 \times S^1$ with warped product metric $g_M = g_1 + e^{2\varphi} d\theta^2$, then Ricci flow on M, $\frac{\partial g_M}{\partial t} = -2Ric_M$ becomes

$$\begin{cases} \frac{\partial g_1}{\partial t} &= -2Ric_1 + 2d\varphi \otimes d\varphi \\ \varphi_t &= \Delta\varphi \end{cases} \tag{7}$$

so a special case of harmonic-Ricci flow.

Properties of harmonic-Ricci flow

Theorem 14 (Muller, '12)

- *The solution of* (6) *exists for a short time and is unique.*
- *If* $\alpha(t)$ *is positive and non-increasing and* $e^{-\phi}$ *solves adjoint heat equation* $\left(-\frac{\partial}{\partial t} - \Delta + R - \alpha|df|^2\right) e^{-\phi} = 0$ *then the functional*

$$\mathcal{F}(f, g, \phi) = \int_M (R_g + |\nabla\phi|_g^2 - \alpha|df|_g^2)e^{-\phi}dV_g$$

  is non-decreasing. In fact, (f, g) *can be interpreted as gradient flow of* $\mathcal{F}$ *for particularly chosen* ϕ*.*
- *If* $\alpha(t) \geq \alpha_0 > 0$ *and* $|df|^2(x_k, t_k) \to \infty$ *as* $t_k \to T$*, then* $R(x_k, t_k) \to \infty$ *as well.*
- *If* $0 < \alpha_0 \leq \alpha(t) \leq \alpha_1 < \infty$ *and* $T < \infty$ *is maximal singular time. Then*

$$\limsup_{t \nearrow T} \left(\max_{x \in M} |Rm(x, t)|^2 \right) = \infty.$$

Special case of 2 dimensional domain

Theorem 15 (Buzano-Rupflin, '17)

- *Let (f, g) be solution of (6) and maximal singular time $T < \infty$. Then both map and curvature must blow up:*

$$\limsup_{t \nearrow T} \max_{x \in M} |K_g| = \infty \quad \text{and} \quad \limsup_{t \nearrow T} \max_{x \in M} \frac{1}{2} |df(x,t)|_g^2 = \infty.$$

- *If coupling function $\alpha \in [\alpha_0, \alpha_1]$ satisfies*

$$\alpha_0 > 2 \max\{K_\tau\}$$

where K_τ denotes sectional curvature of N in direction τ, then the solution (f, g) of (6) is smooth for all time.

Remark 2

Even though Ricci flow and harmonic map flow may develop singularities, its coupling system behaves more regularly!

Thank You!