A tour of Sobolev spaces by Muramatu's integral formula

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We consider the L_p Sobolev space of integer order on $\mathbb{R}^n$.

$$W_p^m(\mathbb{R}^n) = \{f \in L_p(\mathbb{R}^n) : \partial^\alpha f \in L_p(\mathbb{R}^n) \quad (|\alpha| \le m)\}$$

with $m \in \mathbb{N}$, $1 \le p \le \infty$.

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Notation

- $\|f\|_p = \|f\|_{L_p(\mathbb{R}^n)}$
- $B(x, r)$ the ball with center x and radius r
- $\phi * f(x) = \langle f, \phi(x - \cdot)\rangle$ for $f \in \mathcal{D}'(\mathbb{R}^n)$, $\phi \in C_0^\infty(\mathbb{R}^n)$
- $\phi * f(x) = \int_{\mathbb{R}^n} \phi(x - y) f(y)\, dx$ for $f \in L_p(\mathbb{R}^n)$
- p' is defined by $\frac{1}{p} + \frac{1}{p'} = 1$ for $1 \leq p \leq \infty$
- $A \lesssim B$ means $A \leq CB$ with some constant C.
- ∂^k denotes one of $\partial^\alpha = (\frac{\partial}{\partial x})^\alpha$ with $|\alpha| = k$ for $k \in \mathbb{N}$.
- $\partial_j = \frac{\partial}{\partial x_j}$ with $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$
- $\nabla^m f = (\partial^\alpha f)_{|\alpha|=m}$; $\nabla^m f \in X$ means $\partial^\alpha f \in X$ for all α, $|\alpha| = m$; $\|\nabla^m f\|_X = \sum_{|\alpha|=m} \|\partial^\alpha f\|_X$
- For $K(x)$ and $t > 0$ we set $K_t(x) := t^{-n} K(x/t)$.

Muramatu's integral formula

Take $\varphi \in C_0^\infty(B(0,1))$ with $\int_{\mathbb{R}^n} \varphi(x)\,dx = 1$. For $0 < \epsilon < R$ and $f \in \mathcal{D}'(\mathbb{R}^n)$

$$\varphi_R * f(x) - \varphi_\epsilon * f(x) = \int_\epsilon^R \frac{\partial}{\partial t}\{\varphi_t * f(x)\}\,dt = -\int_\epsilon^R M_t * f(x)\,\frac{dt}{t}$$

with $M(x) = \sum_{j=1}^n \partial_j(x_j\varphi(x))$, since $\frac{\partial}{\partial t}\varphi_t(x) = -t^{-1}M_t(x)$. Letting $\epsilon \to 0^+$ gives

$$f = \int_0^R M_t * f\,\frac{dt}{t} + \varphi_R * f.$$

It is convenient that $M(x)$ can be written as a sum of derivatives of higher order. Let $\rho \in C_0^\infty(B(0,1))$ with $\int_{\mathbb{R}^n} \rho(x)\,dx = 1$, and set, for a given $N \in \mathbb{N}$,

$$\varphi(x) = \sum_{|\beta|<N} \frac{1}{\beta!}\{x^\beta \rho(x)\}.$$

Then

$$f = \int_0^R M_t * f \, \frac{dt}{t} + \varphi_R * f \qquad \text{with } M(x) = \sum_{|\beta|=N} \frac{N}{\beta!} \partial^\beta \{x^\beta \rho(x)\}. \tag{M0}$$

Observe $(\partial^\alpha K)_t * f = t^{|\alpha|} K_t * (\partial^\alpha f)$ and $\|\varphi_R * f\|_\infty \leq R^{-n/p} \|f\|_p$.

Theorem (Muramatu's integral formula)

For $f \in \mathcal{D}'(\mathbb{R}^n)$ and $m \in \mathbb{N}$ there exist C^∞ functions φ and K_j $(j = 1, \ldots, n)$ supported on $B(0,1)$ such that $\int_{\mathbb{R}^n} K_j(x)\, dx = 0$ and

$$f = \int_0^R \sum_{j=1}^n t^m (K_j)_t * (\partial_j^m f) \, \frac{dt}{t} + \varphi_R * f. \qquad \text{(M2: two-term version)}$$

Moreover, if $f \in L_p(\mathbb{R}^n)$ with $1 \leq p < \infty$ then

$$f = \int_0^\infty \sum_{j=1}^n t^m (K_j)_t * (\partial_j^m f) \, \frac{dt}{t}. \qquad \text{(M1: one-term version)}$$

Muratatu's approach

- Muramatu mainly used

$$f = \int_0^R \frac{dt}{t} \int_0^t M_t * \tilde{M}_s * f \, \frac{ds}{s} + \int_0^R M_t * \varphi_t * f \, \frac{dt}{t} + \varphi_R * f, \qquad (1)$$

  which is obtained by substituting the RHS of (1) into f in the integral.
- Sobolev and Besov spaces of fractional order
- A general domain $\Omega \subset \mathbb{R}^n$ Remark. (M3) should be adjusted to Ω.
- $f \in W_p^m(\Omega)$ is characterized by $t^{-m} M_t * f(x) \in L_p(\Omega, L_2((0,1), \frac{dt}{t}))$.

Our approach

- (M1) and (M2) are main tools.
- Sobolev spaces of integer order (and partly of fractional order)
- The whole space $\mathbb{R}^n$ (or a special Lipschitz domain)

Differential dimension

When considering embeddings for $W_p^m(\mathbb{R}^n)$, the quantity $m - n/p$ plays an important role. We call it the differential dimension of $W_p^m(\mathbb{R}^n)$.

If we set $f_\lambda(x) = f(\lambda x)$ for $\lambda > 0$, then

$$\|f_\lambda\|_{W_p^m} = \sum_{|\alpha| \le m} \lambda^{|\alpha| - n/p} \|\partial^\alpha f\|_p$$
$$= \lambda^{m - n/p} \sum_{|\alpha| = m} |\partial^\alpha f\|_p + \text{small order} \quad \text{as } \lambda \to \infty.$$

space	differential dimension
$W_p^m(\mathbb{R}^n)$	$m - n/p$
$L_q(\mathbb{R}^n)$	$-n/q$
$\mathcal{C}^\sigma(\mathbb{R}^n)$	σ

Roughly speaking, an embedding $X \subset Y$ holds when the differential dimension of X is larger than or equal to that of Y.

List of Theorems in Sobolev spaces

1. $W_p^m(\mathbb{R}^n) \subset L_q(\mathbb{R}^n)$ for $m - n/p > -n/q$ or $m - n/p = -n/q$
2. $W_p^m(\mathbb{R}^n) \subset \mathcal{C}^\sigma(\mathbb{R}^n)$ for $\sigma = m - n/p > 0$
3. $W_p^m(\mathbb{R}^n)$ for $m - n/p = 0$ Trudinger's inequality
4. $W_p^m(\mathbb{R}^n) \subset BMO$ or VMO for $m - n/p = 0$ (omitted)
5. Gagliardo-Nirenberg inequality and its generalization
$$f \in L_q,\ \nabla^m f \in L_p \Longrightarrow \nabla^k f \in L_r$$
6. Breziz-Gallout-Wainger inequality
7. Brezis-Wainger inequality: $W_p^{m+1}(\mathbb{R}^n)$ for $m - n/p = 0$
8. Trace theorem (omitted)
9. Complex interpolation $[L_p(\mathbb{R}^n), W_p^m(\mathbb{R}^n)]_\theta = W_p^k(\mathbb{R}^n)$ with $k = m\theta$, $0 < \theta < 1$ (omitted)
10. Real interpolation $(L_p(\mathbb{R}^n), W_p^m(\mathbb{R}^n))_{\theta,q} = B_{pq}^{m\theta}(\mathbb{R}^n)$ with $0 < \theta < 1$, $1 \leq q \leq \infty$ (omitted)

Poofs by Muramatu's integral formula

Theorem (Sobolev inequality for simple cases)

Let $m \in \mathbb{N}$, $1 \le p < q \le \infty$, $m - n/p > -n/q$. *Then*

$$W_p^m(\mathbb{R}^n) \subset L_q(\mathbb{R}^n)$$

$$\|f\|_q \le C(n,m,p,q)\|f\|_p^{1-\frac{n}{m}(\frac{1}{p}-\frac{1}{q})}\|\nabla^m f\|_p^{\frac{n}{m}(\frac{1}{p}-\frac{1}{q})}$$

Proof (Muramatu 1975). We use (M2) $f = \int_0^R \sum_{j=1}^n t^m (K_j)_t * (\partial_j^m f)\,\frac{dt}{t} + \varphi_R * f$.
Observe $\|K_t * g\|_q \le \|K_t\|_u \|g\|_p = t^{-n(1-1/u)}\|K\|_u\|g\|_p$ if $\frac{1}{p} + \frac{1}{u} = 1 + \frac{1}{q}$.

$$\begin{aligned}\|f\|_q &\lesssim \int_0^R t^{m-(\frac{n}{p}-\frac{n}{q})}\|\nabla^m f\|_p \frac{dt}{t} + R^{-(\frac{n}{p}-\frac{n}{q})}\|f\|_p \\ &\lesssim R^{m-\frac{n}{p}+\frac{n}{q}}\|\nabla^m f\|_p + R^{-\frac{n}{p}+\frac{n}{q}}\|f\|_p.\end{aligned}$$

Set $R^m = \|f\|_p / \|\nabla^m f\|_p$. □

Theorem (Embeddings into Hölder-Zygmund spaces)

Let $m \in \mathbb{N}$, $1 \le p < \infty$, $\sigma = m - n/p > 0$. *Then*

$$W_p^m(\mathbb{R}^n) \subset \mathcal{C}^\sigma(\mathbb{R}^n)$$

$$\|f\|_\infty \le C(n,m,p)\|f\|_p^{1-\frac{n}{mp}}\|\nabla^m f\|_p^{\frac{n}{mp}}$$

$$[f]_\sigma \le C(n,m,p)$$

We define the difference operator Δ_h by $\Delta_h f(x) = f(x+h) - f(x)$, and $[f]_\sigma := \sup_{x,h} |\Delta_h f(x)|/|h|^\sigma$ for $0 < \sigma < 1$, and $[f]_1 := \sup_{x,h} |\Delta_h^2 f(x)|/|h|$, etc.

Proof (Muramatu 1975). $f \in L_\infty(\mathbb{R}^n)$ follows from Sobolev inequality for simple cases.

Case 1: $0 < \sigma < 1$. (M1) gives

$$\Delta_h f = \int_0^\infty \sum_{j=1}^n t^m (\Delta_h K_j)_t * (\partial_j^m f)\,\frac{dt}{t}. \qquad \text{(M1')}$$

Since $\|\Delta_h K_t\|_{p'} \leq 2\|K_t\|_{p'}$ and $\Delta_h K_t(x) = \int_0^1 t^{-n}(h/t) \cdot \nabla K(\frac{x+\theta h}{t})\, d\theta$, we have $\|(\Delta_h K_t) * g\|_\infty \leq \|\Delta_h K_t\|_{p'} \|g\|_p \lesssim \min\{t^{-n/p}, t^{-n/p}\frac{|h|}{t}\}\|g\|_p$.
Then the change of variables $t = |h|s$ gives

$$|\Delta_h f| \lesssim \int_0^\infty t^{m-n/p} \min\{1, \frac{|h|}{t}\}\|g\|_p \frac{dt}{t} \lesssim |h|^\sigma \|g\|_p \int_0^\infty s^\sigma \min\{1, s^{-1}\} \frac{ds}{s}.$$

Case 2: $\sigma = 1$. The proof goes in the same as in case 1 except that we use $\|\Delta_h^2 K_t\|_{p'} \lesssim \min\{1, (|h|/t)^2\}$ instead of $\|\Delta_h K_t\|_{p'}$.

Case 3: $\sigma > 1$. Apply the results of case 1 and case 2 to $\partial^\alpha f$ with $|\alpha| < m$. □

Theorem (Sobolev inequality for $m - n/p = -n/q$)

Let $m \in \mathbb{N}$, $1 \leq p < q < \infty$, $m - n/p = -n/q$. Then

$$W_p^m(\mathbb{R}^n) \subset L_q(\mathbb{R}^n),$$
$$\|f\|_q \leq C(n,m,p)\|\nabla^m f\|_p.$$

Muramatu's method is to use the estimate $|f(x)| \lesssim \int_{\mathbb{R}^n} |x-y|^{m-n}|\nabla^m(y)|\,dy$ and the Hardy-Littlewood-Sobolev (HLS) inequality for the Riesz potential. Here we give a proof of incorporating the method of Hedberg(1972) who derived the HLS inequality. Recall that the Hardy-Littlewood maximal function Mf is defined by

$$Mf(x) = \sup_{B \ni x} \frac{1}{|B|}\int_B |f(y)|\,dy, \qquad (B\text{: balls.})$$

Proof. We use (M1): $f = \int_0^\infty \sum_{j=1}^n t^m (K_j)_t * (\partial_j^m f)\, \frac{dt}{t}$.
Evaluate $K_t * g(x) = \int_{\mathbb{R}^n} K_t(x-y) g(y)\, dy$ with $g = |\nabla^m f|$ in two ways.

$$|K_t * g(x)| \leq t^{-n} \|K\|_\infty \int_{|x-y|<t} |g(y)|\, dy \leq |B(0,1)| \|K\|_\infty Mg(x).$$

Hölder's inequality gives $|K_t * g(x)| \leq \|K_t\|_{p'} \|g\|_p = t^{-n/p} \|K\|_{p'} \|g\|_p$. Then

$$|f| \lesssim \int_0^R t^m Mg \,\frac{dt}{t} + \int_R^\infty t^{m-n/p} \|g\|_p \,\frac{dt}{t} \lesssim R^{n/p-n/q} Mg + R^{-n/q} \|g\|_p.$$

Choosing R so that $R^{n/p} = \|g\|_p / Mg$, we get $|f|^q \lesssim \|g\|^{p-q} (Mg)^p$. The theorem follows from the L_p boundedness of M for $1 < p < \infty$.
For $p = m = 1$ from the fact that $M : L_1 \to$ weak-L_1 it follwos that $\lambda^q \int_{|f(x)|>\lambda} 1\, dx \lesssim \int_{\mathbb{R}^n} |\nabla f(x)|\, dx$. Apply this to $f_k(x) = (|f(x)| - 2^k)_+ \wedge 2^k$ with $k \in \mathbb{Z}$. (the details and the case $p = 1$, $m \geq 2$ are omitted. cf. Saloff-Coste 2002) □

Theorem (Refined Sobolev inequality for $m - n/p = -n/q$)

Let $m \in \mathbb{N}$, $1 < p < q < \infty$, $m - n/p = -n/q$. If $f \in \mathcal{S}'(\mathbb{R}^n)$ satisfies $\nabla^m f \in L_p$ and $f \in \dot{B}^{-n/q}_{\infty\infty}(\mathbb{R}^n)$, then $f \in L_q(\mathbb{R}^n)$ and

$$\|f\|_q \leq C(n,p,q)\|\nabla^m f\|_p^{p/q}\|f\|_{\dot{B}^{-n/q}_{\infty\infty}}^{1-p/q}.$$

Here $\dot{B}^{-s}_{\infty\infty}(\mathbb{R}^n) = \dot{B}^{-s}$ with $s > 0$, which is called the homogeneous Besov space, is defined by the heat kernel:

$$f \in \dot{B}^{-s} \quad \Longleftrightarrow \quad \sup_{t>0} t^s \|G_t * f\|_\infty := \|f\|_{\dot{B}^{-s}} < \infty,$$

where $G_t(x) = (4\pi t^2)^{-n/2} \exp(-|x|^2/4t^2)$.

It is easy to see that the refined Sobolelv inequality implies the standard one:

$$\|G_t * f\|_\infty \leq \|G_t\|_{q'}\|f\|_q = t^{-n/q}\|G\|_{q'}\|f\|_q \quad \Longrightarrow \quad \|f\|_{\dot{B}^{-n/q}} \lesssim \|f\|_q. \quad \square$$

cf. Ledoux(2003) proved for $1 \le p < \infty$ and $m = 1$, assuming $f \in W^m_p(\mathbb{R}^n)$ in addition to $\nabla^m f \in L_p$. Dao-Lam-Lu (2021) proved using the heat equation.
Proof of the refined Sobolev inequality: There exist functions $\{\eta^k\}$ such that

$$\varphi = \sum_{k=0}^{\infty} \eta^k * G_{2^{-k}}$$

and that $\|\eta^k\|_1 \le C(N)2^{-kN}$ for any $N > 0$ (see Stein's book, Chap III, Sec. 1.3). We use (M2) and

$$\begin{aligned} \|\varphi_R * f\|_\infty &\le \sum_{k=0}^{\infty} \|(\eta^k)_R * G_{2^{-k}R} * f\|_\infty \le \sum_{k=0}^{\infty} \|\eta^k\|_1 \|G_{2^{-k}R} * f\|_\infty \\ &\lesssim \sum_{k=0}^{\infty} 2^{-kN}(2^{-k}R)^{-n/q} \|f\|_{\dot{B}^{-n/q}}. \end{aligned}$$

Then $|f| \lesssim R^{n/p-n/q} M[|\nabla^m f|] + R^{-n/q}\|f\|_{\dot{B}^{-n/q}}$. □

Theorem (Trudinger's inequality (Ozawa 1995) (cf. Strichartz 1972))

Let $m \in \mathbb{N}$, $1 < p < \infty$, $m - n/p = 0$. *Set*

$$\Phi_p(t) = \exp(t) - \textstyle\sum_{k \in \mathbb{N} \cup \{0\},\, k < p-1} \frac{1}{k!} t^k.$$

Then there exist $C = C(n,p)$ *and* $c = c(n,p)$ *such that for* $f \in W_p^m(\mathbb{R}^n)$ *with* $f \neq 0$

$$\int_{\mathbb{R}^n} \Phi_p\left(c \left(\frac{|f(x)|}{\|\nabla^m f\|_p} \right)^{p/(p-1)} \right) dx \le C \left(\frac{\|f\|_p}{\|\nabla^m f\|_p} \right)^p.$$

In the same as the usual proof we use the Sobolev inequality in the embedding $W_p^m(\mathbb{R}^n) \subset L_q(\mathbb{R}^n)$ with $m - n/p = 0 > -n/q$. The proof given previously yields

$$\|f\|_q \le C(n,p) q \|f\|_p^{n/mq} \|\nabla^m f\|_p^{1-n/mq}.$$

We need a better estimate concerning q.

Lemma (Sobolev inequality with optimal constant (Ozawa 1995))

Let $m \in \mathbb{N}$, $1 < p \le q < \infty$, $m - n/p = 0$. *Then* $f \in W_p^m(\mathbb{R}^n)$ *satisfies*

$$\|f\|_q \le C(n,p) q^{1/p'} \|f\|_p^{p/q} \|\nabla^m f\|_p^{1-p/q}.$$

(cf. Kozono-Wadade 2008)

Proof of Lemma. Setting $H_j(x) = \int_0^R t^m (K_j)_t(x) \frac{dt}{t}$, we rewrite (M2) as

$$f = \sum_{j=1}^{n} H_j * (\partial_j^m f) + \varphi_R * f. \qquad \text{(M2-b)}$$

Since $|H_j(x)| \le C(n,p)|x|^{m-n}$ for $|x| < R$, and $|H_j(x)| = 0$ for $|x| \ge R$. Then

$$\|f\|_q \lesssim \left(1 + \frac{q}{p'}\right)^{1/q + 1/p'} \|f\|_p^{p/q} \|\nabla^m f\|_p^{1-p/q}. \qquad \square$$

Proof. Trudinger's inequality follows by applying this lemma with $q = kp'$ to evaluate $\sum_{k=1}^{\infty} \frac{1}{k!} (c\|f\|_p)^{kp'}$. $\square$

Theorem (Classical Gagliardo-Nirenberg (GN) ineqaulity)

Let $k \in \mathbb{N}$, $m \in \mathbb{N}$, $1 \le k < m$, $1 < p \le \infty$, $1 < q \le \infty$. Define $1 < r \le \infty$ by

$$\frac{1}{r} = \frac{k}{mp} + \frac{m-k}{mq}, \quad i.e. \quad k - \frac{n}{r} = \left(\frac{k}{m}\right)\left(m - \frac{n}{p}\right) + \left(1 - \frac{k}{m}\right)\left(-\frac{n}{q}\right)$$

If $f \in L_q(\mathbb{R}^n)$ and $\nabla^m f \in L_p(\mathbb{R}^n)$, then $\nabla^k f \in L_r(\mathbb{R}^n)$ and

$$\|\nabla^k f\|_r \le C(n, m, k, p, q)\|f\|_q^{1-k/m}\|\nabla^m f\|_p^{k/m}.$$

Proof. Applying (M2) to $\partial^k f$ gives

$$\partial^k f = \int_0^R \sum_{j=1}^n t^{m-k}(\partial^k K_j)_t * (\partial_j^m f)\,\frac{dt}{t} + R^{-k}(\partial^k \varphi)_R * f. \tag{M2}$$

Observe $K_t * g(x) \lesssim Mg(x)$ with $K = \partial^k K_j$, $g = \partial_j^m f$.

Hence

$$|\partial^k f| \lesssim \int_0^R t^{m-k} M[|\nabla^m f|] \frac{dt}{t} + R^{-k} Mf \lesssim R^{m-k} M[|\nabla^m f|] + R^{-k} Mf.$$

Choosing R so that $R^m = Mf / M[|\nabla^m f|]$, we get (cf. Maz'ya-Shaposhnikova 1999)

$$|\partial^k f| \lesssim M[|\nabla^m f|]^{k/m} (Mf)^{1-k/m}.$$

By Hölder's inequality and the L_p boudedness of M we have

$$\begin{aligned} \|\partial^k f\|_r^r &\lesssim \int_{\mathbb{R}^n} (M[|\nabla^m f|]^p)^{kr/mp} \left((Mf)^q\right)^{(m-k)r/mq} \, dx \\ &\lesssim \left(\int_{\mathbb{R}^n} M[|\nabla^m f|]^p \, dx \right)^{kr/mp} \left(\int_{\mathbb{R}^n} (Mf)^q \, dx \right)^{(m-k)r/mq} \\ &\lesssim \|\nabla^m f\|_p^{kr/m} \|f\|_q^{(m-k)r/m}. \quad \square \end{aligned}$$

Theorem (Gagliardo-Nirenberg inequality with BMO terms)

When $p = \infty$ or $q = \infty$, the classical Gagliardo-Nirenberg inequality also holds if L_∞-norm is replaced by BMO-norm.

Case 1: $q = \infty$, $1 < p < \infty$. $\qquad \|\partial^k f\|_r \lesssim \|f\|_{BMO}^{1-k/m} \|\nabla^m f\|_p^{k/m}$.

Case 2: $p = \infty$, $1 < q < \infty$. $\qquad \|\partial^k f\|_r \lesssim \|f\|_q^{1-k/m} \|\nabla^m f\|_{BMO}^{k/m}$.

Case 3: $p = q = \infty$. $\qquad \|\partial^k f\|_\infty \lesssim \|f\|_{BMO}^{1-k/m} \|\nabla^m f\|_{BMO}^{k/m}$.

$$\|f\|_{BMO} := \sup_B \frac{1}{|B|} \int_B |f(x) - f_B| \, dx, \qquad f_B = \frac{1}{|B|} \int_B f(x) \, dx,$$

where the supremum is taken over all balls B.
Meyer-Rivière(2003): case 1 for $m = 2$, $k = 1$. Strzelecki(2006): case 1 for general m, k with the additional assumption $f \in W_p^m(\mathbb{R}^n)$.

Proof. We can prove these inequalities by slightly modifying the proof for the classical GN inequality. Instead of $K_t * g(x) \lesssim Mg(x)$ we use

$$|K_t{*}g(x)| = \left| \int_{\mathbb{R}^n} K_t(y)\{g(x-y) - g_{B(x,t)}\}\, dy \right| \lesssim \|g\|_{BMO} \quad \text{if } \int_{\mathbb{R}^n} K_t(y)\, dy = 0. \qquad \square$$

Theorem (GN inequality with the homogeneous Besov norm (Dao et al 2021))

The GN inequality with BMO terms also holds if BMO-norm is replaced by $\dot{B}^0_{\infty\infty}$-norm. (*Note that* $BMO \subset \dot{B}^0_{\infty\infty}$.)

Dao-Lam-Li (2021) proved this theorem using the heat equation.

Proof. Case 1. We can give a proof by (M2) with $R^{-k}(\partial^k\varphi)_R * f = \varphi_R * (\partial^k f)$. We have $\|G_t * \partial^k f\|_\infty \lesssim t^{-k}\|\partial^k f\|_{\dot{B}^{-k}} \lesssim t^{-k}\|f\|_{\dot{B}^0}$, since $f \in \dot{B}^0$ implies $\partial^k f \in \dot{B}^{-k}$.

$$\begin{aligned} \|\varphi_R * (\partial^k f)\|_\infty &\le \textstyle\sum_{j=0}^\infty \|(\eta^j)_R * G_{2^{-j}R} * (\partial^k f)\|_\infty \\ &\lesssim \textstyle\sum_{j=0}^\infty 2^{-jN}(2^j R)^{-k}\|f\|_{\dot{B}^0} \lesssim R^{-k}\|f\|_{\dot{B}^0}. \qquad \square \end{aligned}$$

Theorem (Brezis-Gallouet-Wainger (BGW) inequality; BG 1980, BW 1980)

Let $m \in \mathbb{N}$, $k \in \mathbb{N}$, $1 \le p < \infty$, $1 < q < \infty$, $m - n/p > 0$ and $k - n/q = 0$. If $f \in W_q^k(\mathbb{R}^n)$ and $\nabla^m f \in L_p(\mathbb{R}^n)$, then

$$\|f\|_{L_\infty} \le C(n, m, k, p, q)(1 + \|f\|_{W_q^k} \log(e + \|\nabla^m f\|_p))^{1/q'}.$$

Proof (cf. Ozawa 1995). We use (M2):

$f = \int_0^R \sum_{j=1}^n t^m (K_j)_t * (\partial_j^m f) \frac{dt}{t} + \varphi_R * f$. For $q \le r < \infty$

$$\begin{aligned} |f| &\lesssim \int_0^R t^{m-n/p} \|\nabla^m f\|_p \frac{dt}{t} + R^{-n/r} \|f\|_r \\ &\lesssim R^\sigma \|\nabla^m f\|_p + R^{-n/r} r^{1-1/q} \|f\|_{W_q^k} \end{aligned}$$

with $\sigma = m - n/p$.

Set $R = (e + \|\nabla^m f\|_p)^{-1/\sigma}$ and $r = q \log(e + \|g\|_p)$. Then

$$|f| \lesssim 1 + e^{n/q\sigma} q^{1/q'} (\log(e + \|g\|_p))^{1/q'} \|f\|_{W_q^k}. \quad \square$$

Theorem (Brezis-Wainger (BW) inequality — almost Lipschitz; BW 1980)

Let $m \in \mathbb{N}$, $1 < p < \infty$, $m - n/p = 0$. *If* $f \in W_p^{m+1}(\mathbb{R}^n)$, *then*

$$|\Delta_h f(x)| \le C(n, m, p)\|f\|_{W_p^{m+1}} |h|(1 + \log_+ |h|^{-1})^{1/p'}.$$

Here $\log_+ s = \max\{\log s, 0\}$ *for* $s > 0$.

Proof(cf. Ozawa 1995). Let $p \le q < \infty$. By (M2) with m replaced by $m+1$

$$\Delta_h f = \int_0^R \sum_{j=1}^n t^{m+1}(\Delta_h (K_j)_t) * (\partial_j^{m+1} f) \frac{dt}{t} + \Delta_h(\varphi_R * f).$$

Observe $\|\Delta_h (K_j)_t\|_{p'} \le 2\|(K_j)_t\|_{p'} = 2t^{-n/p}\|K_j\|_{p'} = 2t^{-m}\|K_j\|_{p'}$ and

$$|\Delta_h(\varphi_R * f)| \le |h|\|\nabla(\varphi_R * f)\|_\infty \le |h|\|\varphi_R\|_{q'}\|\nabla f\|_q \lesssim |h| R^{-n/q} q^{1/p'}\|f\|_{W_p^{m+1}}.$$

Then

$$|\Delta_h f| \lesssim \left(R + |h| R^{-n/q} q^{1/p'}\right) \|f\|_{W_p^{m+1}}.$$

When $|h| \leq e^{-p}$, setting $R = |h|$ and $q = -\log|h|$ gives

$$|\Delta_h f| \lesssim |h| \left(1 + e^n (\log|h|^{-1})^{1/p'}\right) \|f\|_{m+1,p}.$$

When $|h| > e^{-p}$, setting $R = |h|$ and $q = p$ gives

$$|\Delta_h f| \lesssim |h| \left(1 + e^n p^{1/p'}\right) \|f\|_{m+1,p}.$$

□

Fractional Sobolev spaces

Let $m > 0$, $1 < p < \infty$. $\mathcal{F}f = \hat{f}$ denotes the Fourier transform of $f \in \mathcal{S}' = \mathcal{S}'(\mathbb{R}^n)$.

$$W_p^m(\mathbb{R}^n) := \{f \in \mathcal{S}' : \mathcal{F}^{-1}[(1 + |\xi|^2)^{m/2}\hat{f}(\xi)] \in L_p(\mathbb{R}^n)\}.$$

Recall (M0) $f = \int_0^R M_t * f \, \frac{dt}{t} + \varphi_R * f$ with $M = \sum_{|\beta|=N} (\sqrt{-1}\partial)^\beta (\rho^\beta)$.
For $f \in \mathcal{S}(\mathbb{R}^n)$ set $(-\Delta)^{m/2} f = \mathcal{F}^{-1}[|\xi|^m \hat{f}(\xi)]$ and write

$$\begin{aligned} M_t * f(x) &= \mathcal{F}^{-1}\Big[\sum_{|\beta|=N} (t\xi)^\beta \mathcal{F}\rho^\beta(t\xi) \cdot \hat{f}(\xi)\Big](x) \\ &= t^m \mathcal{F}^{-1}\Big[\sum_{|\beta|=N} |t\xi|^{-m} (t\xi)^\beta \mathcal{F}\rho^\beta(t\xi) \cdot |\xi|^m \hat{f}(\xi)\Big](x) \\ &= t^m K_t * (-\Delta)^{m/2} f(x) \end{aligned}$$

with $K = \mathcal{F}^{-1}[\sum_{|\beta|=N} |\xi|^{-m} \xi^\beta \mathcal{F}\rho^\beta(\xi)] \in C^\infty(\mathbb{R}^n) \cap L_1(\mathbb{R}^n) \cap L_\infty(\mathbb{R}^n)$.

Thus we get

$$f = \int_0^R t^m K_t * (-\Delta)^{m/2} f \, \frac{dt}{t} + \varphi_R * f.$$

$K(x)$ satisfies

- $\int_{\mathbb{R}^n} K_t(x)\, dx = 0$,
- $|K(x)| \lesssim (1 + |x|)^{-n-(N-m)}$,
- $|K_t * (-\Delta)^{m/2} f| \lesssim M[(-\Delta)^{m/2} f]$,
- $|K_t * (-\Delta)^{m/2} f| \lesssim t^{-n/p} \|K\|_{p'} \|(-\Delta)^{m/2} f\|_p$.

These properties enable us to deal with fractional Sobolev spaces.

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