The Bernoulli-type free boundary problem and its application

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A joint work with Jianfeng Cheng (SCU) and Zhouping Xin (CUHK).

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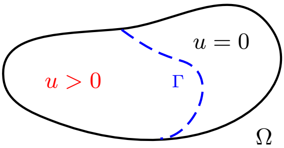
June 20, 2022.

Outlines

1. The Bernoulli-type free boundary problem

The Bernoulli-type problem: A free boundary problem with a transition condition across the free boundary.

$$\begin{cases} \Delta u = 0, & \text{in } \Omega \cap \{u > 0\}, \\ u = g, & \text{on } \partial\Omega, \\ u = 0, \quad \dfrac{\partial u}{\partial n} = |\nabla u| = \Lambda(x) & \text{on } \Gamma = \Omega \cap \partial\{u > 0\}, \end{cases}$$



where $g(x)$ is given boundary function, Γ is the unknown free boundary, $\Lambda(x)$ is given gradient function on free boundary.

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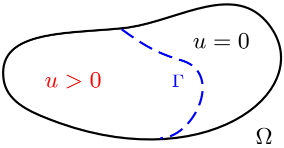

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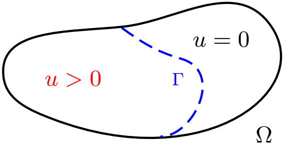

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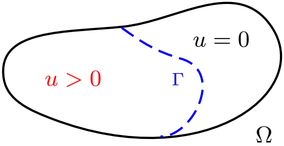

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- A harmonic function simultaneously satisfies linear homogenous Dirichlet and inhomogeneous Neumann boundary conditions on the free boundary.
- A early example came from Bernoulli's law and the constant-pressure condition in the study of steady waves in hydrodynamics and Stokes waves.
- Because of the jump in the gradient across the free boundary, the optimal possible regularity is Lipschitz, i.e. $u \in C^{0,1}(\Omega)$.

Variational Problem.

$$\min_{\mathcal{K}} \int_{\Omega} |\nabla u|^2 + \Lambda(x)\chi_{\{u>0\}} dx, \quad \text{for } \mathcal{K} = \left\{ u \in W^{1,2}(\Omega) | u = g \text{ on } \partial\Omega \right\}.$$

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- u is subharmonic in Ω, i.e. $\Delta u \geq 0$ in Ω in distributional sense.
- Optimal regularity of the solution.

$$u \in C^{0,1}(\Omega).$$

- Chapter 3-4 in
  L. Caffarelli, S. Salsa, A geometric approach to free boundary problems, *Graduate Studies in Mathematics*, 68, American Mathematical Society, Providence, RI, 2005.

Regularity of the free boundary

1. Non-degenerate point $|\nabla u| = \Lambda(x) \geq \Lambda_0 > 0$.

Linear case. $\Delta u = 0$, the free boundary Γ is analytic.

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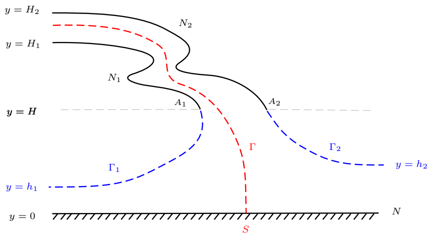
Answer: Singular !
If it is C^1-smooth, Hopf boundary point lemma implies that $\dfrac{\partial u}{\partial n} \neq 0.$

- A. Friedman, Variational principles and free-boundary problems, A Wiley-Interscience Publication. Pure and Applied Mathematics, *John Wiley & Sons, Inc., New York*, 1982.
- E. Shargorodsky, J. F. Toland, Bernoulli free-boundary problems, *Mem. Amer. Math. Soc.*, 196, no. 914, (2008).
- A. Figalli, H. Shahgholian, An overview of unconstrained free boundary problems, *Philos. Trans. Roy. Soc. A*, 373, (2015).

2. An application: steady impinging jet flows with gravity

Physical problem: Steady two-dimensional free-surface flows of an inviscid and incompressible fluid emerging from a nozzle and falling under gravity.



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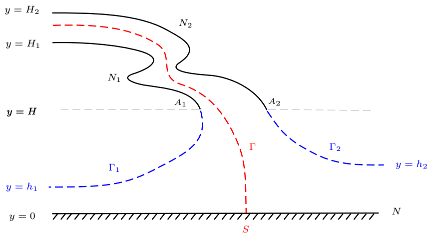

Numerical results:

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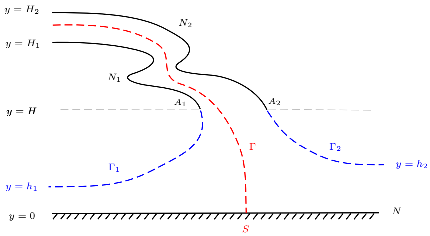

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Applications:

- Vertical /Short Takeoff and Landing (V/STOL) Aircraft.
- Terrestrial rocket launch.
- Fabricating glassy metals.
- ......

Incompressible ideal irrotational flows ingravity field.

$$\begin{cases} div\vec{u} = 0, \\ \vec{u} \cdot \nabla \vec{u} + \nabla p = g e_n \\ \nabla \times \vec{u} = 0. \end{cases}$$

Mathematical foundations

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Stream function formulation.

$$\begin{cases} \psi_{xx} + \psi_{yy} = 0, \quad \psi_x = -v, \ \psi_y = u, & \text{for 2D plane flows,} \\ \psi_{rr} + \psi_{yy} - \frac{1}{r}\psi_r = 0, \quad \frac{\psi_r}{r} = -v, \ \frac{\psi_y}{r} = u, & \text{for 3D axially symmetric flows.} \end{cases}$$

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Bernoulli's law.

$$\frac{1}{2}|\vec{u}|^2 + p + gy = \text{constant} := \mathcal{B}.$$

Constant pressure condition. Given the atmospheric pressure p_{atm} on Γ, ie.,

$$p = p_{atm} \quad \text{on } \Gamma.$$

Bernoulli-type free boundary condition.

$$\begin{cases} |\nabla\psi| = \sqrt{2(\mathcal{B} - p_{atm} + gy)} & \text{on } \Gamma, \quad \text{for 2D plane flows,} \\ \frac{1}{r}|\nabla\psi| = \sqrt{2(\mathcal{B} - p_{atm} + gy)} & \text{on } \Gamma, \quad \text{for 3D axially symmetric flows.} \end{cases}$$

Mathematical results on jets under gravity

- **Under gravity:** Alt-Caffarelli-Friedman, **J. Reine Angew Math.,** 1982. Incompressible inviscid irrotational flow falling from the semi-infinitely long nozzle with infinite height.

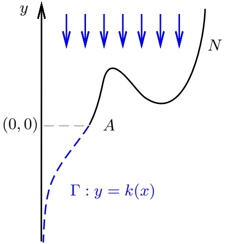


Figure: 3D axially symmetric jet

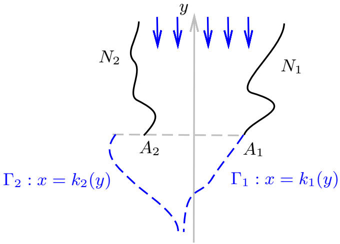


Figure: 2D asymmetric jet

Main difficulties.

"The mathematical literature on jets with gravity is **very meager***. The reason for this is that the hodograph method which has been successfully used in steady 2-dimensional problems for jets and cavities without gravity cannot be extended to the case where gravity is present."*

Alt-Caffarelli-Friedman, Jet flows with gravity, *J. Reine Angew. Math.*, (1982).

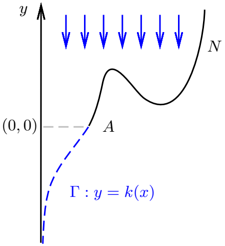
Axially symmetric nozzle with infinite height. N is a x-graph.

Bernoulli-type FBP.

$$\begin{cases} \psi_{rr} + \psi_{yy} - \frac{1}{r}\psi_r = 0, & \text{in } \Omega \cap \{0 < \psi < Q\}, \\ \psi = 0, \quad \text{on axis}, & \psi = Q, \quad \text{on } N, \\ \frac{1}{r}|\nabla\psi| = \sqrt{2(\mathcal{B} - p_{atm} + gy)} & \text{on } \Gamma. \end{cases}$$

Result. For any flux $Q > 0$, there exists a unique solution (ψ, Γ) to the jet problem with gravity which satisfies that there exists a unique $\mathcal{B}$, such that Γ initiates smoothly at A and

$$k(y) = \frac{\sqrt{2Q}}{|y|^{1/4}}(1 + o(1)) \qquad \text{as} \quad y \to -\infty.$$



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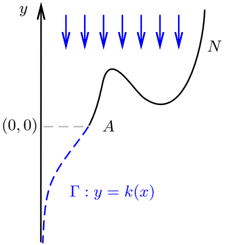

Remark. $k(0) = 1$ is so-called continuous fit condition, which implies that the FB initiates at A.

2D asymmetric nozzle with infinite height.

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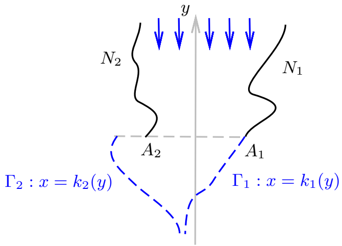

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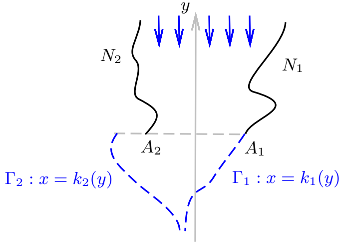

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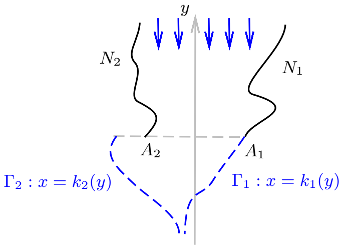

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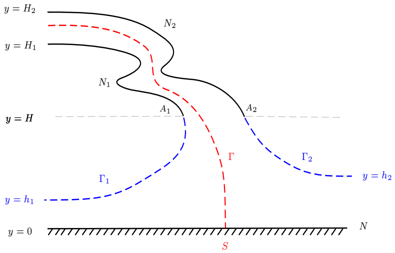
Ill-posedness result. In general, one can **NOT** fit continuously at both endpoints.

Aim: To establish the well-posedness theory on incompressible impinging jet with two asymmetric free boundaries issuing from a nozzle **with finite height**.



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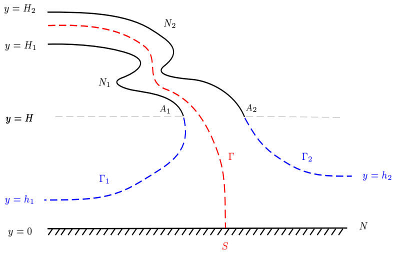
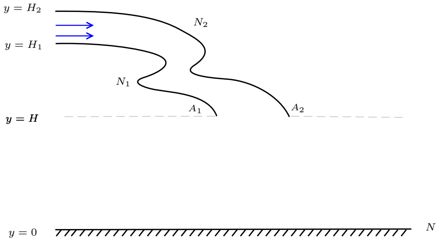


Question: Given a total mass flux Q in the infinite inlet and an atmosphere pressure p_{atm}, does there exist an impinging jet under gravity with two continuous fit conditions ?

Notations and assumptions.



Ground $N : y = 0$.

Left nozzle wall $N_1 : x = g_1(y)$ for $y \in [H, H_1]$.

Right nozzle wall $N_1 : x = g_2(y)$ for $y \in [H, H_2]$.

H : the distance between orifice and the ground.

$A_1 = (-1, H)$, $A_2 = (1, H)$: the endpoints of the nozzle.

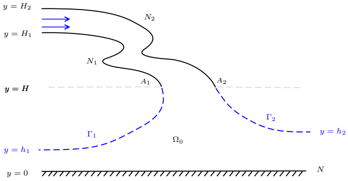
Assumptions: $g_1(y), g_2(y) \in C^{2,\alpha}$, $g_1(y) < g_2(y)$, $\lim\limits_{y \to H_i^-} g_i(y) = -\infty$ for $i = 1, 2$.

Notations:

Q : the total incoming flux in upstream,

Q_1 : the effluent flux in negative x-direction,

Q_2 : the effluent flux in positive x-direction.



Facts:

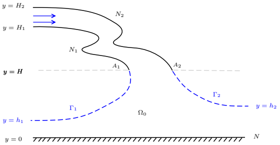
- $Q = Q_1 + Q_2$.
- Q_1 and Q_2 are unknown apriori.

A solution to the impinging jet flow problem

Definition. A vector $(u, v, p, \Gamma_1, \Gamma_2)$ is called a solution to the impinging jet problem, provided that

(1) The free streamlines Γ_1 and Γ_2 can be described by C^1-smooth functions $x = k_1(y)$ and $x = k_2(y)$, respectively, and there exist two constants $h_1, h_2 \in (0, H)$, such that

$$\lim_{y \to h_1^+} k_1(y) = -\infty \quad \text{and} \quad \lim_{y \to h_2^+} k_2(y) = +\infty.$$



(2) The free boundaries Γ_1 and Γ_2 are analytic, and satisfy

$$g_1(H) = k_1(H) = -1 \quad \text{and} \quad g_2(H) = k_2(H) = 1, \tag{1}$$

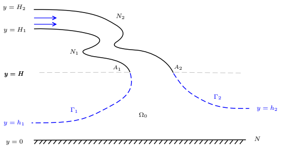
and

$$g_1'(H + 0) = k_1'(H - 0) \quad \text{and} \quad g_2'(H + 0) = k_2'(H - 0). \tag{2}$$

(3) $(u, v, p) \in \left(C^{1,\alpha}(\Omega_0) \cap C^0(\overline{\Omega}_0)\right)^3$ solves the steady incompressible Euler system in gravity field.

(4) $p = p_{atm}$ on Γ_1 and Γ_2.

Remark. The constants h_1 and h_2 are indeed the **asymptotic widths** of the impinging jet in left and right downstream. The property 1 in the definition implies that the free boundaries Γ_1 and Γ_2 can not oscillate in downstream.



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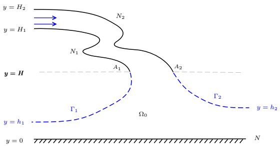

Remark. The conditions (1) and (2) are so-called *continuous fit conditions* and *smooth fit conditions* for the impinging jet, which mean that the free boundaries connect the nozzle walls smoothly.

Q: How to guarantee the continuous fit conditions ?

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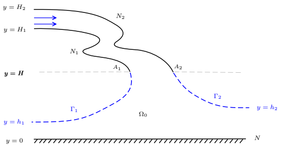


Remark. The conditions (1) and (2) are so-called *continuous fit conditions* and *smooth fit conditions* for the impinging jet, which mean that the free boundaries connect the nozzle walls smoothly.

Q: How to guarantee the continuous fit conditions ?

Remark. Bernoulli's law and constant pressure condition give that

$$\frac{u^2+v^2}{2}+gy=\mathcal{B}-p_{atm} \qquad \text{on} \quad \Gamma_1\cup\Gamma_2.$$

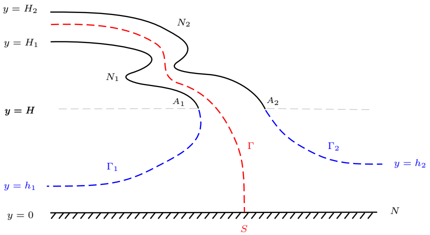
The Bernoulli's constant $\mathcal{B}$ is undetermined at the present stage. We treat $(\mathcal{B},Q_1)$ as a pair of parameter in solving the impinging jet problem, and show that there exists a pair of $(\mathcal{B},Q_1)$ to guarantee the continuous fit conditions.

Theorem 1. (Cheng-Du-Xin, arXiv:1808.01494). For any given atmosphere pressure p_{atm} and total incoming flux $Q > 2\sqrt{gH^3}$, there exists a solution $(u, v, p, \Gamma_1, \Gamma_2)$ to the incompressible impinging jet problem with gravity. Furthermore,

(1). $v < 0$ in $\Omega_0 \cup \Gamma_1 \cup \Gamma_2$.

(2). There exists a unique smooth streamline $\Gamma : x = k(y)$, which separates the impinging jet with two different downstream, and Γ goes to the inlet of the nozzle and intersects the ground N at the unique point S. Moreover,

$$k'(0+0) = 0.$$



Main results

Theorem 2. (Cheng-Du-Xin, arXiv:1808.01494). Assume that there exists a large $R_0 > 1$, such that the nozzle wall N_i can be described by $y = g_i^{-1}(x)$ for $x < -R_0$, $i = 1, 2$, then

$$(u, v, p) \to \left(-\frac{Q_1}{h_1}, 0, p_1(y)\right), \ \nabla(u, v) \to 0 \text{ and } \ \nabla p \to (0, -g) \tag{3}$$

uniformly in any compact subset of $(0, h_1)$ as $x \to -\infty$, and

$$(u, v, p) \to \left(\frac{Q - Q_1}{h_2}, 0, p_2(y)\right), \ \nabla(u, v) \to 0 \text{ and } \ \nabla p \to (0, -g) \tag{4}$$

uniformly in any compact subset of $(0, h_2)$ as $x \to +\infty$, where

$$\frac{Q_1^2}{h_1^2} + 2gh_1 = \frac{(Q - Q_1)^2}{h_2^2} + 2gh_2, \ p_i(y) = p_{atm} + g(h_i - y) \text{ for } y \in (0, h_i).$$

Similarly, in upstream,

$$(u, v, p) \to \left(\frac{Q}{H_2 - H_1}, 0, p_0(y)\right), \ \nabla(u, v) \to 0 \text{ and } \ \nabla p \to (0, -g) \tag{5}$$

uniformly in any compact subset of (H_1, H_2) as $x \to -\infty$, where

$$p_0(y) = p_{atm} + g(h_1 - y) + \frac{Q_1^2}{2h_1^2} - \frac{Q^2}{2(H_1 - H_2)^2} \text{ for } y \in (H_1, H_2).$$

$$\textbf{Total flux condition } Q > 2\sqrt{gH^3}. \tag{6}$$

The total flux condition (6) guarantees the following important facts.

Relationship between $(\mathcal{B}, Q_1)$ and (h_1, h_2).

Fact 1. The parameter $(\mathcal{B}, Q_1)$ can be determined uniquely by h_1 and h_2.

$$Q_1 = \sqrt{2(\mathcal{B} - p_{atm} - gh_1)}h_1 \quad \text{and} \quad Q - Q_1 = \sqrt{2(\mathcal{B} - p_{atm} - 2gh_2)}h_2. \tag{7}$$

Fact 2. The asymptotic heights h_1 and h_2 can be determined uniquely by $\mathcal{B}$ and Q_1.

Fact 3. The asymptotic height h_1 is monotone increasing with respect to Q_1, and the asymptotic height h_2 is monotone decreasing with respect to Q_1, for any fixed $\mathcal{B}$ and Q.

Comments on the results

Total flux condition $Q > 2\sqrt{gH^3}$. (6)

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Remark. Unfortunately, the condition (6) prevents us to obtain the existence of jet under gravity without the horizontal plate N as $H \to \infty$.

Comments on the results

Remark. The critical assumption $Q > 2\sqrt{gH^3}$ also guarantees that there does not exist the stagnation point in the fluid domain, especially on the free boundary. The advantage of this fact lies in exclusion the possible singularity on the free boundary of water wave with gravity. The singularity of the free surface flows with gravity is closely related to a very interesting problem, the so-called **Stokes Conjecture:**

G. G. Stokes, (1880), at any stagnation point the free surface has a symmetric corner of $\frac{2\pi}{3}$.

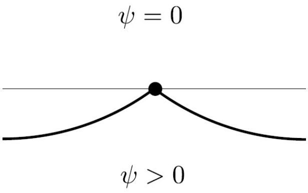


Figure: Stokes corner

Amick-Fraenkel-Toland, On the Stokes conjecture for the wave of extreme form, **Acta Math.**, (1982).

Varvaruca-Weiss, A geometric approach to generalized Stokes conjectures, **Acta Math.**, (2011).

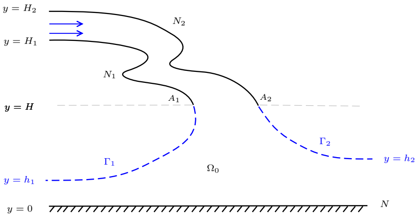
Varvaruca-Weiss, The Stokes conjecture for waves with vorticity, **Ann. I. H. Poincaré-AN**, (2012).

Step 1. Mathematical setting and free boundary problem.

Stream function $\psi_y = u$ and $\psi_x = -v$, the following **Bernoulli-type free boundary problem** as follows,

$$\begin{cases} \Delta\psi = 0 \text{ in } \ \Omega \cap \{0 < \psi < Q\}, \\ \dfrac{\partial\psi}{\partial n} = \sqrt{2\lambda - 2gy} \ \text{ on } \ \Gamma_1, \ \dfrac{\partial\psi}{\partial n} = -\sqrt{2\lambda - 2gy} \ \text{ on } \ \Gamma_2, \\ \psi = 0 \text{ on } N_1 \cup \Gamma_1, \ \psi = Q \text{ on } N_2 \cup \Gamma_2, \ \ \psi = Q_1 \text{ on } N, \end{cases} \tag{8}$$

where $\lambda = \mathcal{B} - p_{atm}$.



Ω: **the possible fluid field**,

$\Omega_0 = \Omega \cap \{0 < \psi < Q\}$: **fluid field**,

$\Gamma_1 : \Omega \cap \partial\{\psi > 0\}$, $\Gamma_2 : \Omega \cap \partial\{\psi < Q\}$,

$\Gamma : \Omega \cap \partial\{\psi = Q_1\}$.

- A free boundary problem with two undetermined parameters (λ, Q_1).

Step 2. Bernoulli's law $\Longrightarrow \lambda = \frac{Q_1^2}{2h_1^2} - gh_1 = \frac{(Q-Q_1)^2}{2h_2^2} - gh_2$,
the condition $Q > 2\sqrt{gH^3}$ guarantees the monotonicity of λ with respect to h_1 and h_2, we obtain a lower bound of λ as

$$\lambda \geq \frac{(\max\{Q_1, Q - Q_1\})^2}{2H^2} + gH \geq \frac{Q^2}{8H^2} + gH.$$

Hence, we will solve the Bernoulli-type free boundary problem (8) for any parameters $\lambda \geq \frac{Q^2}{8H^2} + gH$ and $Q_1 \in [0, Q]$ first.

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Step 3. Solvability of Bernoulli-type free boundary problem (Variational approach).
Introduce a functional and an admissible set

$$J_\lambda(\psi) = \int_\Omega |\nabla\psi|^2 + (2\lambda - 2gy)\chi_{\{0<\psi<Q\}\cap D} dxdy, \tag{9}$$

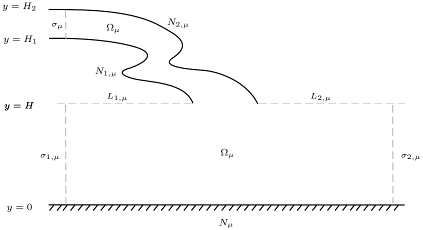
and

$$K_{Q_1} = \{\psi \in H_{loc}(\mathbb{R}^2) \,|\psi = Q_1 \text{ lies below } N, \psi = Q \text{ lies in the right of } N_2 \cup L_2,$$
$$\psi = 0 \text{ lies in the left of } N_1 \cup L_1\}.$$

where χ_E is the characteristic function of a set E and $D = \{0 < y < H\}$.

Since $J_\lambda(\psi)=+\infty$ for any $\psi\in K_{Q_1}$, and we have to truncate the domain Ω

$$
\begin{aligned}
&H_{1,\mu}=\max\{y\mid g_1(y)=-\mu\},\\
&H_{2,\mu}=\min\{y>H_{1,\mu}\mid g_2(y)=-\mu\},\\
&N_{1,\mu}=N_1\cap\{x\geq-\mu\},\\
&N_{2,\mu}=N_2\cap\{x\leq\mu\},\\
&L_{1,\mu}=L_1\cap\{x\geq-\mu\},\\
&L_{2,\mu}=L_2\cap\{x\leq\mu\}.
\end{aligned}
$$

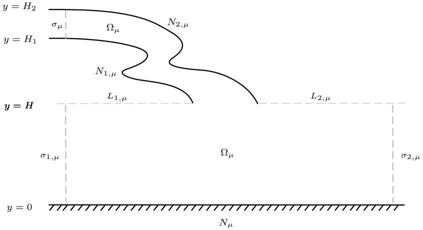


$$\sigma_{1,\mu}=\{x=-\mu,0\leq y\leq H\},\quad \sigma_{2,\mu}=\{x=\mu,0\leq y\leq H\},$$
$$N_\mu=N\cap\{-\mu\leq x\leq\mu\},\quad \sigma_\mu=\{x=-\mu,H_{1,\mu}\leq y\leq H_{2,\mu}\}.$$

Ω_μ is bounded by $N_{1,\mu},N_{2,\mu},L_{1,\mu},L_{2,\mu},\sigma_\mu,\sigma_{1,\mu},\sigma_{2,\mu}$ and N_μ.

Since $J_\lambda(\psi) = +\infty$ for any $\psi \in K_{Q_1}$, and we have to truncate the domain Ω

$$
\begin{aligned}
H_{1,\mu} &= \max\{y \mid g_1(y) = -\mu\}, \\
H_{2,\mu} &= \min\{y > H_{1,\mu} \mid g_2(y) = -\mu\}, \\
N_{1,\mu} &= N_1 \cap \{x \geq -\mu\}, \\
N_{2,\mu} &= N_2 \cap \{x \leq \mu\}, \\
L_{1,\mu} &= L_1 \cap \{x \geq -\mu\}, \\
L_{2,\mu} &= L_2 \cap \{x \leq \mu\}.
\end{aligned}
$$



$$\sigma_{1,\mu} = \{x = -\mu, 0 \leq y \leq H\}, \quad \sigma_{2,\mu} = \{x = \mu, 0 \leq y \leq H\},$$

$$N_\mu = N \cap \{-\mu \leq x \leq \mu\}, \quad \sigma_\mu = \{x = -\mu, H_{1,\mu} \leq y \leq H_{2,\mu}\}.$$

Ω_μ is bounded by $N_{1,\mu}, N_{2,\mu}, L_{1,\mu}, L_{2,\mu}, \sigma_\mu, \sigma_{1,\mu}, \sigma_{2,\mu}$ and N_μ.

Additional Boundary conditions.

$$\psi = \max\left\{-\sqrt{2\lambda - 2gh_1}y + Q_1, 0\right\} \quad \text{on} \quad \sigma_{1,\mu},$$

$$\psi = \min\left\{\sqrt{2\lambda - 2gh_2}y + Q_1, Q\right\} \quad \text{on} \quad \sigma_{2,\mu},$$

$$\psi = \frac{(y - H_{1,\mu})Q}{H_{2,\mu} - H_{1,\mu}} \quad \text{on} \quad \sigma_\mu.$$

Define a truncated functional as

$$J_{\lambda,\mu}(\psi) = \int_{\Omega_\mu} |\nabla\psi|^2 - (2\lambda - 2gy)\chi_{\{\psi<Q\}\cap D_\mu} dxdy, \tag{10}$$

where $D_\mu = \{-\mu < x < \mu, 0 < y < H\}$.

Truncated variational problem $(P_{\lambda,Q_1,\mu})$**:** For any $\mu > 1$, $Q_1 \in [0, Q]$ and $\lambda \geq \frac{Q^2}{8H^2} + gH$, find a $\psi_{\lambda,Q_1,\mu} \in K_{\lambda,Q_1,\mu}$, such that

$$J_{\lambda,\mu}(\psi_{\lambda,Q_1,\mu}) = \min_{\psi \in K_{\lambda,Q_1,\mu}} J_{\lambda,\mu}(\psi),$$

where the admissible set

$$K_{\lambda,Q_1,\mu} = \{\psi \in K_{Q_1} \mid \psi \text{ satisfies the additional boundary conditions}\}.$$

The existence of minimizer $\psi_{\lambda,Q_1,\mu}$ follows from the standard variational method.

Some important facts.

(1) $\Delta\psi_{\lambda,Q_1,\mu} = 0$ in $\Omega_\mu \cap \{0 < \psi_{\lambda,Q_1,\mu} < Q\}$. Furthermore, $\psi_{\lambda,Q_1,\mu} \in C^{0,1}(\Omega_\mu)$ and $\psi_{\lambda,Q_1,\mu} \in C^{2,\alpha}(D)$ for any compact subset D of $\Omega_\mu \cap \{0 < \psi_{\lambda,Q_1,\mu} < Q\}$.

(2) $0 \leq \psi_{\lambda,Q_1,\mu} \leq Q$ in Ω_μ and $0 < \psi_{\lambda,Q_1,\mu} < Q$ in $\Omega_\mu \cap \{y > H\}$.

(3) The free boundaries $\Gamma_{1,\lambda,Q_1,\mu}$ and $\Gamma_{2,\lambda,Q_1,\mu}$ are analytic.

(4) $|\nabla\psi_{\lambda,Q_1,\mu}| = \sqrt{2\lambda - 2gy}$ on $\Gamma_{1,\lambda,Q_1,\mu} \cup \Gamma_{2,\lambda,Q_1,\mu}$.

Alt-Caffarelli, Existence and regularity for a minimum problem with free boundary, **J. Reine Angew. Math.**, (1981).

Step 4. Properties of the minimizer $\psi_{\lambda,Q_1,\mu}$.

Step 4.1.

$$\max\left\{-\sqrt{2\lambda-2gh_1}y+Q_1,0\right\}\leq\psi_{\lambda,Q_1,\mu}\leq\min\left\{\sqrt{2\lambda-2gh_2}y+Q_1,Q\right\}, \qquad (11)$$

and

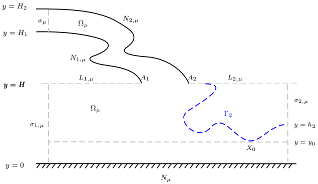
$$\max\left\{\frac{(y-H_{1,\mu})Q}{H_{2,\mu}-H_{1,\mu}},0\right\}<\psi_{\lambda,Q_1,\mu}(x,y)<Q \quad \text{in} \quad \Omega_\mu\cap\{y>H\}. \qquad (12)$$

Remark. The bound (11) gives that

$$0<-\sqrt{2\lambda-2gh_1}y+Q_1\leq\psi_{\lambda,Q_1,\mu}(x,y) \text{ for } y<h_1,$$

and

$$\psi_{\lambda,Q_1,\mu}(x,y)\leq\sqrt{2\lambda-2gh_2}y+Q_1<Q \text{ for } y<h_2,$$



which implies that the free boundary $\Gamma_{i,\lambda,Q_1,\mu}$ lies upon the asymptotic height h_i.

Step 4. Properties of the minimizer $\psi_{\lambda,Q_1,\mu}$.

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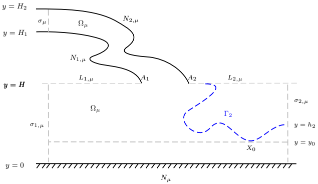

which implies that the free boundary $\Gamma_{i,\lambda,Q_1,\mu}$ lies upon the asymptotic height h_i.

Step 4.2. Monotonicity $\psi_{\lambda,Q_1,\mu}$ is increasing with respect to x. $\Longrightarrow v\le 0$ in $\bar{\Omega}_\mu$.

Step 4. Properties of the minimizer $\psi_{\lambda,Q_1,\mu}$.

Step 4.1.

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Remark. The bound (11) gives that

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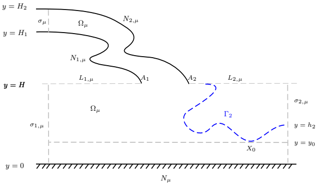

which implies that the free boundary $\Gamma_{i,\lambda,Q_1,\mu}$ lies upon the asymptotic height h_i.

Step 4.2. Monotonicity $\psi_{\lambda,Q_1,\mu}$ is increasing with respect to x. $\Longrightarrow v\leq 0$ in $\bar{\Omega}_\mu$.

Step 4.3. Uniqueness $\psi_{\lambda,Q_1,\mu}$ is unique for any fixed λ, Q_1 and μ.

Step 5. Properties of free boundaries $\Gamma_{i,\lambda,Q_1,\mu}$.

Step 5.1. The monotonicity of $\psi_{\lambda,Q_1,\mu}$ with respect to x gives that there exists a function $k_{i,\lambda,Q_1,\mu}(y)$, such that

$$D_\mu \cap \{0 < \psi_{\lambda,Q_1,\mu} < Q\} = D_\mu \cap \{k_{1,\lambda,Q_1,\mu}(y) < x < k_{2,\lambda,Q_1,\mu}(y)\}.$$

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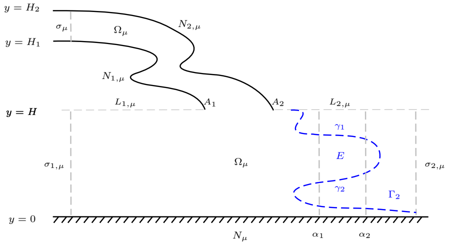
Step 5.2. Non-Oscillation lemma

Notations:
End points of $\gamma_i : (\alpha_i, \beta_i), (\alpha_i, \eta_i)$, for $i = 1, 2$.

Non-Oscillation lemma:
$|\alpha_1 - \alpha_2| \leq C \max\{|\beta_1 - \beta_2|, |\eta_1 - \eta_2|\}$.

(width $\leq C$ height)



Remark. The non-oscillation lemma implies that the existence of $\lim_{y \to y_0} k_{i,\lambda,Q_1,\mu}(y)$ for any $y \in (h_i, H]$.

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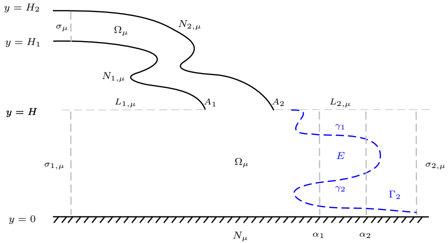

Remark. The non-oscillation lemma implies that the existence of $\lim_{y \to y_0} k_{i,\lambda,Q_1,\mu}(y)$ for any $y \in (h_i, H]$.

Step 5.3. The continuity of $k_{i,\lambda,Q_1,\mu}(y)$.

Step 6. Almost continuous fit conditions.

For any $\mu > 1$, $\bar{\lambda}_\mu > \dfrac{Q^2}{8H^2} + gH$ and there exists a $\bar{Q}_{1,\mu} \in [0, Q]$, such that

(1) $k_{1,\bar{\lambda}_\mu,\bar{Q}_{1,\mu},\mu}(H) \leq -1$ and $k_{2,\bar{\lambda}_\mu,\bar{Q}_{1,\mu},\mu}(H) \geq 1$.

(2) $k_{1,\bar{\lambda}_\mu,\bar{Q}_{1,\mu},\mu}(H) = -1$ or $k_{2,\bar{\lambda}_\mu,\bar{Q}_{1,\mu},\mu}(H) = 1$.

(3) $k_{2,\bar{\lambda}_\mu,\bar{Q}_{1,\mu},\mu}(H) = 1$ for $\bar{Q}_{1,\mu} < Q$.

(4) $k_{1,\bar{\lambda}_\mu,\bar{Q}_{1,\mu},\mu}(H) = -1$ for $\bar{Q}_{1,\mu} > 0$.

Step 6. Almost continuous fit conditions.

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(2) $k_{1,\bar{\lambda}_\mu,\bar{Q}_{1,\mu},\mu}(H) = -1$ or $k_{2,\bar{\lambda}_\mu,\bar{Q}_{1,\mu},\mu}(H) = 1$.
(3) $k_{2,\bar{\lambda}_\mu,\bar{Q}_{1,\mu},\mu}(H) = 1$ for $\bar{Q}_{1,\mu} < Q$.
(4) $k_{1,\bar{\lambda}_\mu,\bar{Q}_{1,\mu},\mu}(H) = -1$ for $\bar{Q}_{1,\mu} > 0$.

Remark. Obviously, as long as the critical cases $\bar{Q}_{1,\mu} = 0$ and $\bar{Q}_{1,\mu} = Q$ are excluded, we can obtain the continuous fit conditions

$$k_{1,\bar{\lambda}_\mu,\bar{Q}_{1,\mu},\mu}(H) = -1 \text{ and } k_{2,\bar{\lambda}_\mu,\bar{Q}_{1,\mu},\mu}(H) = 1.$$

Step 6. Almost continuous fit conditions.

For any $\mu > 1$, $\bar{\lambda}_\mu > \dfrac{Q^2}{8H^2} + gH$ and there exists a $\bar{Q}_{1,\mu} \in [0, Q]$, such that

(1) $k_{1,\bar{\lambda}_\mu,\bar{Q}_{1,\mu},\mu}(H) \leq -1$ and $k_{2,\bar{\lambda}_\mu,\bar{Q}_{1,\mu},\mu}(H) \geq 1$.

(2) $k_{1,\bar{\lambda}_\mu,\bar{Q}_{1,\mu},\mu}(H) = -1$ or $k_{2,\bar{\lambda}_\mu,\bar{Q}_{1,\mu},\mu}(H) = 1$.

(3) $k_{2,\bar{\lambda}_\mu,\bar{Q}_{1,\mu},\mu}(H) = 1$ for $\bar{Q}_{1,\mu} < Q$.

(4) $k_{1,\bar{\lambda}_\mu,\bar{Q}_{1,\mu},\mu}(H) = -1$ for $\bar{Q}_{1,\mu} > 0$.

Remark. Obviously, as long as the critical cases $\bar{Q}_{1,\mu} = 0$ and $\bar{Q}_{1,\mu} = Q$ are excluded, we can obtain the continuous fit conditions

$$k_{1,\bar{\lambda}_\mu,\bar{Q}_{1,\mu},\mu}(H) = -1 \text{ and } k_{2,\bar{\lambda}_\mu,\bar{Q}_{1,\mu},\mu}(H) = 1.$$

Step 6.1. $\psi_{\lambda,\mu}$ and $k_{i,\lambda,Q_1,\mu}(y)$ are continuous dependence with respect to two parameters λ and Q_1.

Step 6.2. Monotonicity

$\psi_{\lambda,Q_1,\mu}(x,y) \geq \psi_{\lambda,Q_1',\mu}(x,y)$ for any $Q_1 > Q_1'$.

This fact implies that the free boundary $x = k_{i,\lambda,Q_1,\mu}(y)$ is decreasing with respect to Q_1.

Step 6.3. Define a set

$$\Sigma_\mu = \{\lambda \mid \text{ there exists a } Q_1 \in (0, Q)\text{, such that } k_{1,\lambda,Q_1,\mu}(H) < -1 \text{ and } k_{2,\lambda,Q_1,\mu}(H) > 1.\}$$

Fact. If $Q_1 = \frac{Q}{2}$, there exists a $\lambda > \frac{Q^2}{8H^2} + gH$, such that if $\lambda - \frac{Q^2}{8H^2} - gH$ is small, then

$$k_{1,\lambda,Q_1,\mu}(H) < -1 \text{ and } k_{2,\lambda,Q_1,\mu}(H) > 1.$$

This fact implies that the set Σ_μ is non-empty.

Step 6.3. Define a set

$$\Sigma_\mu = \{\lambda \mid \text{ there exists a } Q_1 \in (0, Q), \text{ such that } k_{1,\lambda,Q_1,\mu}(H) < -1 \text{ and } k_{2,\lambda,Q_1,\mu}(H) > 1.\}$$

Fact. If $Q_1 = \dfrac{Q}{2}$, there exists a $\lambda > \dfrac{Q^2}{8H^2} + gH$, such that if $\lambda - \dfrac{Q^2}{8H^2} - gH$ is small, then

$$k_{1,\lambda,Q_1,\mu}(H) < -1 \text{ and } k_{2,\lambda,Q_1,\mu}(H) > 1.$$

This fact implies that the set Σ_μ is non-empty.

Step 6.4. The set Σ_μ is uniformly bounded for any $\mu > 1$.

Step 6.3. Define a set

$$\Sigma_\mu = \{\lambda \mid \text{ there exists a } Q_1 \in (0,Q), \text{ such that } k_{1,\lambda,Q_1,\mu}(H) < -1 \text{ and } k_{2,\lambda,Q_1,\mu}(H) > 1.\}$$

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$$k_{1,\lambda,Q_1,\mu}(H) < -1 \text{ and } k_{2,\lambda,Q_1,\mu}(H) > 1.$$

This fact implies that the set Σ_μ is non-empty.

Step 6.4. The set Σ_μ is uniformly bounded for any $\mu > 1$.

Fact. If $Q_1 \in (0,Q)$, there exists a C_0 (independent of μ and Q_1), such that

$$k_{1,\lambda,Q_1,\mu}(H) > -1 \text{ or } k_{2,\lambda,Q_1,\mu}(H) < 1,$$

for any $\lambda > C_0$.

Step 6.3. Define a set

$$\Sigma_\mu = \{\lambda \mid \text{ there exists a } Q_1 \in (0, Q), \text{ such that } k_{1,\lambda,Q_1,\mu}(H) < -1 \text{ and } k_{2,\lambda,Q_1,\mu}(H) > 1.\}$$

Fact. If $Q_1 = \frac{Q}{2}$, there exists a $\lambda > \frac{Q^2}{8H^2} + gH$, such that if $\lambda - \frac{Q^2}{8H^2} - gH$ is small, then

$$k_{1,\lambda,Q_1,\mu}(H) < -1 \text{ and } k_{2,\lambda,Q_1,\mu}(H) > 1.$$

This fact implies that the set Σ_μ is non-empty.

Step 6.4. The set Σ_μ is uniformly bounded for any $\mu > 1$.

Fact. If $Q_1 \in (0, Q)$, there exists a C_0 (independent of μ and Q_1), such that

$$k_{1,\lambda,Q_1,\mu}(H) > -1 \text{ or } k_{2,\lambda,Q_1,\mu}(H) < 1,$$

for any $\lambda > C_0$.

Step 6.5. Taking $\bar{\lambda}_\mu = \sup_{\lambda \in \Sigma_\mu} \lambda$, there exists a $\bar{Q}_{1,\mu} \in [0, Q]$, such that the almost continuous fit conditions hold.

Step 7. The existence of the impinging jet flow

Step 7.1. Taking a sequence $\{\mu_k\}$, such that

$$\bar{\lambda}_{\mu_k} \to \lambda, \quad \bar{Q}_{1,\mu_k} \to Q_1 \text{ and } \psi_{\bar{\lambda}_{\mu_k},\bar{Q}_{1,\mu_k}\mu_k} \to \psi_{\lambda,Q_1},$$

and

$$k_{i,\bar{\lambda}_{\mu_k},\bar{Q}_{1,\mu_k},\mu_k}(y) \to k_{i,\lambda,Q_1}(y),$$

with almost continuous fit conditions

(1) $k_{1,\lambda,Q_1}(H) \leq -1$ and $k_{2,\lambda,Q_1}(H) \geq 1$.

(2) $k_{2,\lambda,Q_1}(H) = 1$ for $Q_1 < Q$.

(3) $k_{1,\lambda,Q_1}(H) = -1$ for $Q_1 > 0$.

Step 7.2. Boundedness and continuity of free boundaries $x = k_{i,\lambda,Q_1}(y)$.

Exclude three cases.

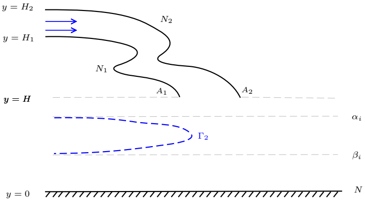

Figure: Case 1

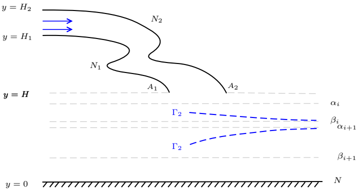

Figure: Case 2

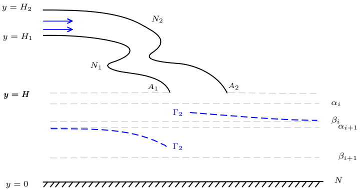

Figure: Case 3

Step 7.3. Exclude the critical cases $Q_1 = 0$ and $Q_1 = Q$ to obtain the continuous fit conditions.

Fact. The possible value Q_1 lies in $(0, Q)$.

Without loss of generality, assume $Q_1 = 0$, we have to exclude the following three cases.

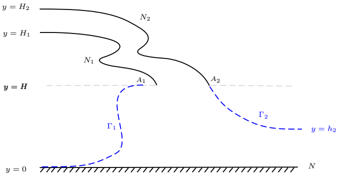

Figure: $k_1(0) = -\infty$

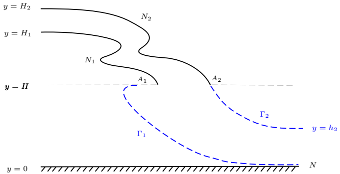

Figure: $k_1(0) = +\infty$

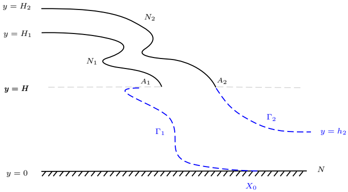

Figure: $k_1(0) = x_0$

Step 8. The existence and properties of the interface $\Gamma : \Omega \cap \{\psi = Q_1\}$.

Fact. Γ can be denoted by $x = k(y)$ for $y \in (0, H_3)$, where $H_3 = \frac{Q_1(H_2 - H_1)}{Q} + H_1$, $\lim_{y \to 0^+} k(y)$ exists and is finite. Moreover, $k'(0+0) = 0$

Thanks !