Effect of the average scalar curvature on Riemannian manifolds

Kwok-Kun Kwong
Australian National University

Reference: Effect of the average scalar curvature on Riemannian manifolds
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Notation: (N^n, g) (complete) Riem mfd

R_{ijji} = sectional curv. $= \langle (\nabla_i \nabla_j - \nabla_j \nabla_i)\partial_j, \partial_i \rangle$ strongest

average

Ricci curvature (Ric) $\quad R_{ij} = \sum_{k,l} g^{kl} R_{kijl}$

average

Scalar curvature $\quad R = tr_g(Ric)$

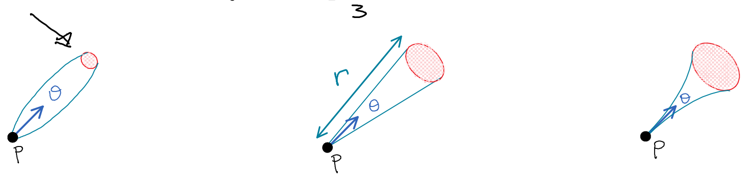
$$= \sum_{i,j} g^{ij} R_{ij}$$

weakest

Main Theme: Roughly, large "curvature" $\Rightarrow$ small "size"

• Geometric meaning of Ric

Area $= c\, r^{n-1} \left(1 - \frac{Ric_p(\theta)}{3} r^2 + O(r^4)\right)$ as $r \to 0$



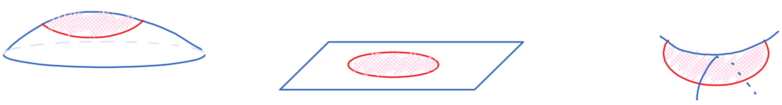
$Ric_p(\theta) > 0$ $\qquad Ric_p(\theta) = 0$ $\qquad Ric_p(\theta) < 0$

• Geometric meaning of R

$|B_r(p)| = b_n r^n \left(1 - \frac{R(p)}{6(n+2)} r^2 + O(r^4)\right)$ as $r \to 0$

Euclid.

(can be taken as def. of $R(p)$)



$R(p) > 0$ $\qquad R(p) = 0$ $\qquad R(p) < 0$

• R tends to oppose $|B_p(r)|$ for small r.

ie. $R(p)>0 \Rightarrow |B_r(p)| < b_n r^n$, r small

- On the other hand,

$$R(p) \geq 0 \not\Rightarrow |B_r(p)| \leq b_n r^n$$

Counter-example

eg. $M = S^2 \times \mathbb{H}^2 \quad \therefore R \equiv 0$

Gray's formula

$$\Rightarrow \quad |B_r(p)| = b_4 r^4 \left(1 + \frac{1}{360 \cdot 4 \cdot 6 \cdot 8}\left(-3|\mathrm{Rm}|^2 + 8|\mathrm{Ric}|^2\right) r^4 + O(r^6)\right).$$

$$= b_4 r^4 \left(1 + \frac{1}{72 \cdot 6 \cdot 8} r^4 + O(r^6)\right) > b_4 r^4$$

close to, but still $>$

Cor: $R \geq 0$ alone is not enough to say something on $|B_p(r)|$, even for small r

- Also, for $n \geq 3$, upper bdd of R is not useful to deduce something topological or geometrical:

$\forall$ mfd $N^{n\geq 3}$ has comp. metric g s.t. $\mathrm{Ric}_g < 0$ (Lohkamp 1994) (cf also Aubin 1970)

- From Gromov's "Four Lectures on Scalar Curvature"

> But this is an illusion:
> there is no single known (are there unknown?) geometric argument, which would make use of this definition. $R(p) = \lim_{r \to 0} \frac{b_n r^n - |B_r(p)|}{C_n r^{n+2}}$
> The immediate reason for this is *the infinitesimal* nature of the volume comparison property: it *doesn't integrate* to the corresponding property of balls of specified, let them be small, radii $r \leq \varepsilon > 0$.[4]
>
> ***Basic Question.*** Does the sign of the scalar curvature have any visible macroscopic effect on the geometry of V? (Say $R \geq k$)

- Situation is quite different for Ric

Bishop-Gromov volume (element) comparison

If $\mathrm{Ric} \geq (n-1)k$

polar coord

Let $dV = \underbrace{F(r,\theta)}_{\text{Volume element / Jacobian}}\, dr\, d\theta$

$$\Rightarrow \quad F(r,\theta) \leq F_k(r) = \begin{cases} r^{n-1} & k=0 \\ \sin^{n-1} r & k=1 \\ \sinh^{n-1} r & k=-1 \end{cases}$$

$\overset{\text{integrate}}{\Rightarrow} \quad |B_r(p)| \leq |B^n_k(r)|$

space form

- Q: Say $\mathrm{Ric} \geq 0$ & $R \geq c > 0$, can we improve

Bishop - Gromov ?

eg. $|B_r(p)| \leq b_n r^n$ − something ?

(I don't know.)

- Gromov conjecture (1986). (N^n, g) complete non-cpt,

  $\mathrm{Ric} \geq 0$, $R \geq n(n-1) \Rightarrow |B_r(p)| \leq C_n r^{n-2}$

  (True under additional "non-collapsing" ass. (B. Zhu 2022))

- But surprisingly, even average $\overline{R}$ can say something about the "size":

<u>Thm</u> (Green, independently by Berger)

(N^n, g) closed, $\overline{R} \geq n(n-1)$

$\Rightarrow$ Conjugate radius $\mathrm{conj}(N) \leq \pi$

$$\left(\begin{array}{l} \mathrm{conj}(N) := \inf_{p \in N} \mathrm{conj}(p) \ \& \\ \mathrm{conj}(p) := \sup\{r : d\exp_p|_v \text{ non-degenerate if } |v| < r\} \end{array} \right)$$

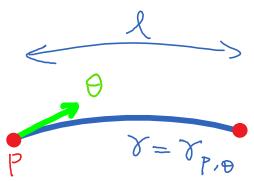
Significance: even $\overline{R}$ can say something "global"!

Key idea: Index lem + Liouville Thm

<u>Pf of Green's Thm</u>

$\ell := \mathrm{Conj}(N)$

Choose $X_i := \sin\left(\frac{\pi t}{\ell}\right) e_i$ along γ $(1 \leq i \leq n-1)$ — parallel, onf



- Index lem:

  $$0 \leq I(X_i, X_i) := \int_0^\ell |\dot{X}_i|^2 - \mathrm{Rm}(\dot{\gamma}, X_i, X_i, \dot{\gamma})\, dt \leftarrow \left(2^{\text{nd}} \text{ var. of } \int |\dot{\gamma}|^2\right)$$

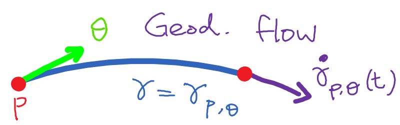
  Index form

  sum $\Longrightarrow$ $$0 \leq (n-1)\frac{\pi^2}{2\ell} - \int_0^\ell \underbrace{\mathrm{Ric}(\dot{\gamma}(t))}_{\mathrm{Ric}(\dot{\gamma},\dot{\gamma})} \sin^2\left(\frac{\pi t}{\ell}\right) dt$$ (as in Bonnet Myers)

Trick: Integrate over $(p, \theta) \in SN$

$$0 \leq (n-1)\frac{\pi^2}{2\ell}\, \omega_{n-1} |N| - \int_{SN} \int_0^\ell \mathrm{Ric}(\dot{\gamma}_{p,\theta}(t)) \sin^2\left(\frac{\pi t}{\ell}\right) dt \, d\mu_{SN}(p,\theta)$$

- Interchange $\int_{SN}$ & $\int_0^\ell$



& geod. flow $\theta \longmapsto \dot{\gamma}_{p,\theta}(t) : SN \to SN$ preserves $d\mu_{SN}$ (Liouville Thm)

$\therefore (n-1)\frac{\pi^2}{2\ell}\, \omega_{n-1} |N| \geq \int_0^\ell \sin^2\left(\frac{\pi t}{\ell}\right) \int_{SN} \mathrm{Ric}(\dot{\gamma}_{p,\theta}(t))\, d\mu_{SN}(p,\theta)\, dt$

$$\therefore (n-1)\frac{\pi^2}{2\ell} \omega_{n-1} |N| \geq \int_0^\ell \sin^2\left(\frac{\pi t}{\ell}\right) \int_{SN} Ric(\dot\gamma_{p,\theta}(t))\, d\mu_{SN}(p,\theta)\, dt$$

$$\overset{\text{Liouville}}{=\!=} \int_0^\ell \sin^2\left(\frac{\pi t}{\ell}\right) \int_{SN} Ric_p(\theta)\, d\mu_{SN}\, dt$$

$$= \frac{\ell}{2} \int_{p\in N} \int_{S_pN} Ric_p(\theta)\, d\theta\, d\mu_N$$

$$= \frac{\ell}{2} \int_N \frac{\omega_{n-1}}{n} R(p)\, d\mu_N$$

$$\geq (n-1)\frac{\ell}{2} \omega_{n-1} |N| \qquad \text{ie. } \ell \leq \pi$$

#

- In view of Green Thm, can $\bar R$ deduce something about geom. of geod. spheres / balls ?

<u>Thm 1</u> (K. 2022) ($k=0$ case)

(N^n, g) closed, $0 \leq Ric \leq \kappa$

$\Rightarrow$ Average vol. of balls

$$\bar V(r) \leq b_n r^n - \frac{b_n \bar R}{\kappa} \int_0^r \left(1 - e^{\frac{\kappa t^2}{6}}\right) t^{n-1}\, dt$$

Equality $\Leftrightarrow$ N is flat & $r \leq inj(N)$

- Similar result if $0 \leq Ric_k := Ric - (n-1)kg \leq \kappa$ as well, which leads to

<u>Cor</u> $0 \leq Ric_1 \leq \kappa$ ($k=1$ case)

$$\Rightarrow |N| \leq |S^n| - b_n \frac{\bar R_1}{\kappa} \int_0^\pi \left(1 - e^{-\kappa \sigma_1(t)}\right) \sin^{n-1} t\, dt$$

(sharp $\Leftrightarrow$ $N = S^n$)

- Ingredients of Thm 1
  1. Jacobian estimate
  2. "Reverse" Jensen ineq
  3. Liouville Thm

- 1 Jac. est ($k=0$ case)

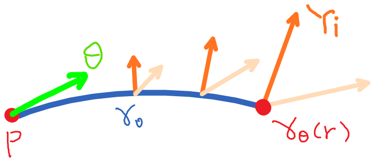
In polar coord around p. (within $\mathrm{inj}(p)$)

$$dV = F(p,r,\theta)\, dr\, d\underbrace{\theta}_{S_pN \cong S^{n-1}}$$

$$\Rightarrow F(p,r,\theta) \le \exp\left[-\int_0^r \int_0^\tau \frac{t^2}{\tau^2} \mathrm{Ric}(\dot\gamma_{p,\theta}(t))\, dt\, d\tau\right] r^{n-1} \quad \text{(no ass. on Ric!)}$$

- **Pf of Jac. est**

(i) Fix r & $\theta \in S_pN$



$e_1, \cdots, e_n = \gamma_0'$ parallel along γ_0

$Y_i(t)$ $(i = 1, \cdots, n-1 = m)$ Jacobi field st

$$\begin{cases} Y_i(0) = 0 \\ Y_i(r) = e_i(r) \end{cases} \Rightarrow F(p,t,\theta) = \frac{\det(Y_1(t), \cdots, Y_m(t))}{\det(\dot Y_1(0), \cdots, \dot Y_m(0))}$$

(ii) Differentiate $\log F$

$$\frac{\partial}{\partial r} \log F(p,r,\theta) = \sum^m \int_0^r \langle \dot Y_i, \dot Y_i \rangle - \langle Rm(Y_i, \dot\gamma)\dot\gamma, Y_i \rangle\, dt$$

$$= \sum^m I(Y_i) \leftarrow \text{index form}$$

(iii) Compare standard Jacobi field

Index lemma: $X(0) = Y(0)$ & $X(r) = Y(r)$

$$\Rightarrow I(\underbrace{Y}_{\text{Jac field}}) \le I(X)$$

Choose $X_i(t) := \frac{t}{r} e_i(t)$

$$\sum^m I(X_i) \xlongequal[\text{compute}]{\text{direct}} -\int_0^r \frac{t^2}{r^2} \mathrm{Ric}(\dot\gamma(t))\, dt + \frac{d}{dr}(\log r^{n-1})$$

ie. $$\frac{\partial}{\partial r} \log F(p,r,\theta) \le \frac{d}{dr}(\log r^{n-1}) - \int_0^r \frac{t^2}{r^2} \mathrm{Ric}(\dot\gamma(t))\, dt \quad \text{—— } (*)$$

- Integrating $(*)$ wrt r & taking exp,

$$\boxed{F(p,r,\theta) \le \exp\left[-\int_0^r \int_0^\tau \frac{t^2}{\tau^2} \mathrm{Ric}(\dot\gamma_{p,\theta}(t))\, dt\, d\tau\right] r^{n-1}}$$

- Next, want to integrate this ineq over $(p,\theta) \in SN$ apply Liouville Thm
  (Green's trick)

  Problem: integrand is $\exp\left[\int \cdots \mathrm{Ric}(\cdots)\right]$ instead of $\int \cdots \mathrm{Ric}(\cdots)$

  Our sol: 2 Reverse Jensen ineq

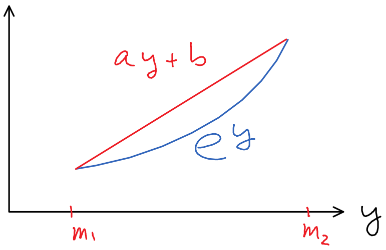
  Lem On probability space (Ω, μ)

  if $m_1 \le f \le m_2$

  $$\Rightarrow \int_\Omega e^f \, d\mu \le a \int_\Omega f \, d\mu + b$$

  where $a = \dfrac{e^{m_2} - e^{m_1}}{m_2 - m_1}$, $b = \dfrac{m_2 e^{m_1} - m_1 e^{m_2}}{m_2 - m_1}$

  Pf



  Convexity

  $\Rightarrow e^y \le ay + b \qquad (m_1 \le y \le m_2)$

  $\Rightarrow$ Integrate!

3 Take $\Omega = \left(SN, \dfrac{d\mu_{SN}}{|N|\,\omega_{n-1}} \right)$,

$f = \mathrm{Ric}(p, \theta) : SN \to \mathbb{R}$

Ass $0 \le \mathrm{Ric} \le K$

Recall: $F(p,r,\theta) \le \exp\left[-\int_0^r \int_0^\tau \frac{t^2}{\tau^2} \mathrm{Ric}(\dot\gamma_{p,\theta}(t))\, dt\, d\tau\right] r^{n-1}$ by 1

- Integrate over $(p,\theta) \in SN$ wrt prob. meas.

$$\bar{A}(r) \le \Big(1 - a \int_0^r \int_0^\tau \frac{t^2}{\tau^2} \underbrace{\fint_{SN} \mathrm{Ric}(\dot\gamma_{p,\theta}(t))\, d\mu_{SN}}_{\frac{\bar{R}}{n} \text{ by Liouville}} \, dt\, d\tau\Big) r^{n-1}$$

average area

$$= \left(1 - \frac{1 - e^{-Kr^2/6}}{nK} \bar{R}\right) \omega_{n-1} r^{n-1}$$

- Integrating again,

$$\bar{V}(r) \le b_n r^n - \frac{b_n \bar{R}}{K} \int_0^r \left(1 - e^{-\frac{Kt^2}{6}}\right) t^{n-1} dt$$

(true also for metric balls of any radius by using cutoff.)

Effect on average mean curvature of geod spheres

• Integrating $(*)$:

$$\frac{\partial}{\partial r}\log F(p,r,\theta) \le \frac{d}{dr}\left(\log r^{n-1}\right) - \int_0^r \frac{t^2}{r^2}\,\mathrm{Ric}(\dot\gamma(t))\,dt$$

over $(p,\theta)\in SN$ & using Green's trick,

$$\Rightarrow \frac{d}{dr}\fint_{SN}\log\left(\frac{F(p,r,\theta)}{r^{n-1}}\right)d\mu_{SN}(p,\theta) \overset{(**)}{\le} -\frac{\overline{R}\,r}{3n}$$

(monotone quantity if $\overline{R}\ge 0$)

Note that there's no curvature ass.! (except $r < \mathrm{inj}(N)$)

• $(**)$ can be reintrepreted as

Thm2 For closed (N,g), if $r<\mathrm{inj}(N)$

$$\Rightarrow \fint_{SN} H(p,r,\theta)\,d\mu_{SN}(p,\theta) \le \frac{n-1}{r} - \frac{\overline{R}\,r}{3n}$$

mc of $S_r(p)$ at $\exp_p(r\theta)$

more generally,

av. of $R - n(n-1)k$

$$\fint_{SN} H(p,r,\theta)\,d\mu_{SN}(p,\theta) \le \frac{F_k'(r)}{F_k(r)} - \frac{\overline{R_k}}{n}\phi_k(r)$$

explicit

• This is surprising to me: [Gray - Vanhecke]

$$\fint_{\mathbb{S}^{n-1}} H(p,r,\theta)d\theta$$
$$=\frac{n-1}{r}-\frac{R(p)}{3n}r-\frac{1}{90n(n+2)}\left(3\|\mathrm{Rm}|_p\|^2+2\|\mathrm{Ric}|_p\|^2+18\Delta R(p)\right)r^3+\underbrace{O(r^5)}_{\text{v. complicated}},$$

for small r

Thm3 (Eigenvalue est.)

space form

(N^n,g) closed, $\mathrm{Ric}_k \ge 0$, $\mathrm{Rm}\le k$, $r<\min\{\mathrm{Inj}(N),\mathrm{Inj}(N_k)\}$

ball in N_k

$$\Rightarrow \min_{p\in N}\lambda_1\left(B_p(r)\right)\le\lambda_1(\mathbb{B}_k(r))-\frac{\overline{R_k}}{n}\frac{\int_0^r\phi(\tau)\,|\phi'(\tau)|\,F_\kappa(\tau)\psi_k(\tau)d\tau}{\fint_N\int_{B(p,r)}\phi(d(p,x))^2d\mu_N(x)d\mu_N(p)}.$$

1st Lap. eigenvalue wrt. Dirichlet cond.

"$=$" N has curv. k.

• This improves Cheng's eigenvalue est.

Questions:

- What does monotonicity (*) mean / imply? (Is it known?)
- How about non-compact (N, g)
  (Some results abt. vol. / area can be extended to
  $|N| < \infty$ & $\int_{SN} \mathrm{Ric}^- < \infty$)
- Other settings? (with boundary, Kahler, Lorentzian $\cdots$)

Thank You!