Some comments on the spectral gap of Schrödinger operators

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The setting for domains

- We study the spectral gap of Schrödinger operators

$$h_L = -\Delta + v$$

  on domains $\Lambda_L = \left(-\frac{L}{2}, +\frac{L}{2}\right)^d$ in $\mathbb{R}^d$ with external and non-negative potentials $v \in L^\infty(\mathbb{R}^d)$.
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- Basic question: How does $\Gamma_v(L)$ behave in the limit $L \to \infty$? How does v influence the asymptotics?

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- A classical question: Does the spectral gap increase/decrease if one adds a potential v to the free Laplacian?
- It turns out that the answer is highly non-trivial and for a generic potential v it is not clear what happens (see papers by Ashbaugh, Benguria, Lavine, and others)

Results in one dimension

- Assuming v decays at least quadratically at infinity one obtains an upper bound, i.e.,

$$\Gamma_v(L) \leq \frac{\alpha}{L^2}$$

  for some $\alpha > 0$ and L large enough.
- The proof is simple and relies on the fact that the second eigenvalue converges to zero like $\sim L^{-2}$ for such potentials.

Results in one dimension: something surprising

- Assuming that v is non-zero and decays faster than $|x|^{-2}$ at infinity, one obtains a surprising result: Namley, one has

$$\lim_{L\to\infty} L^2\Gamma_v(L) = 0 \ .$$

- For example, for potentials $v \in C_0^\infty(\mathbb{R})$, the spectral gap converges to zero strictly faster than for the free Laplacian although the potential is supported on a smaller and smaller fraction of the interval.

Results in one dimension: basic mechanism

- However small the potential, since the first two eigenvalues converge to zero, the potential divides the interval into two "congruent" parts. This leads to an effective decoupling of left- and right-hand side.
- In this way the ground state becomes effectively degenerate in the infinite-volume limit and hence the spectral gap is converging to zero faster than for the free Laplacian.

Results in one dimension: what about a lower bound?

- Question: How fast does the spectral gap converge to zero for fast decaying potentials?
- Conjecture: For compactly supported potentials one expects $\Gamma_v(L) \sim L^{-3}$.

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- Idea: Derive a Harnack inequality for φ_0! (Berhanu & Mohammed, A Harnack Inequality for Ordinary Differential Equations, The Amer. Math. Monthly, 2005)

Results in one dimension: a general lower bound

Theorem (Harnack inequality)

For all $L > 0$ one has

$$\min \varphi_0(x) \geq \mathrm{e}^{-4L\|v\|_{L^1}} \cdot \max \varphi_0(x)$$

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- The key point is to derive a bound on

$$\left| \frac{(\varphi_0)'(x)}{\varphi_0(x)} \right|$$

  using the eigenvalue equation
- Then consider $f(t) := \ln \varphi_0(t(x-y)+y)$ with $t \in [0,1]$ and $x, y \in \Lambda_L$

Results in one dimension: a general lower bound

Theorem (General lower bound)

For all $L > 0$ *and* $v \in L^\infty(\mathbb{R})$ *one has*

$$\Gamma_v(L) \geq \mathrm{e}^{-8L\|v\|_{L^1}} \cdot \frac{\pi^2}{L^2} \; .$$

Results in one dimension: a general lower bound

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- The factor of 8 is not optimal
- The lower bound is very small but holds for general potentials and all L

Results in one dimension: a lower bound for weak compactly supported potentials

- Consider again the Schrödinger operator with Neumann boundary conditions on the interval of length L
- Assume $v(x) = v(-x)$ is a bounded potential with support $[-b, +b]$, $b > 0$. In addition, we shall assume that $\inf v(x) > \gamma > 0$ and $b^2 \|v\|_{L^\infty(\mathbb{R})} < 1/2$.

Results in one dimension: a lower bound for weak compactly supported potentials

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- For such potentials one can prove that

$$\frac{\beta}{L^4} \leq \Gamma_v(L) \leq \frac{\alpha}{L^2} \ ,$$

for some constants $\alpha, \beta > 0$ and all $L > 0$ large enough.

Results in two dimensions

- We first consider potentials $v \in L^{\infty}(\mathbb{R}^2)$ that decay faster than quadratically (we assume Dirichlet boundary conditions from now on)
- The upper bound is again easy and reads αL^{-2} for some $\alpha > 0$.
- But does, like in one dimension, decay the gap faster than for the free Dirichlet Laplacian? Think of compactly supported potentials.

Results in two dimensions

- It turns out that the effect from one dimension disappears! More explicitly, for such potentials one has

$$\frac{\beta}{L^2} \leq \Gamma_v(L) \leq \frac{\alpha}{L^2} \ ,$$

  for some constants $\alpha, \beta > 0$.
- Intuitively, a compactly potential in two dimensions does not lead to an effective decomposition of the domain and therefore not to an effective degeneracy of the ground state in the limiting regime.

A remark on the proof

- A scaling leads to the Dirichlet Laplacian on the domain $\left(-\frac{1}{2},+\frac{1}{2}\right)^2$ with a potential concentrating around zero.
- Then, a result of Ozawa applies (Singular variation of domains and eigenvalues of the Laplacian, Duke Math. J., 1981): the eigenvalues of the Laplacian on a (nice enough) domain with a hole converge to those of the free Laplacian without hole.

Results in higher dimensions

- The same is true for potentials $v \in (L^\infty \cap L^1)(\mathbb{R}^d)$ for $d \geq 3$.

A proof

- The free ground state is given by $\varphi_0 = \left(\frac{2}{L}\right)^{\frac{d}{2}} \prod_{j=1}^{d} \cos\left(\frac{\pi x_j}{L}\right)$.
- Since v is non-negative, one has

$$\frac{(d+3)\pi^2}{L^2} \leq \lambda_1(L) \ .$$

- On the other hand, the variational principle gives

$$\lambda_0(L) \leq \langle \varphi_0, h_L \varphi_0 \rangle \leq \frac{d\pi^2}{L^2} + \|\varphi_0\|_\infty^2 \cdot \|v\|_{L^1(\mathbb{R}^d)} \ .$$

- $\|\varphi_0\|_\infty^2 \sim L^{-d}$ yields the statement.

Results in two dimensions

- Is it possible to construct a potential $v \in L^\infty(\mathbb{R}^2)$ for which one has
$$\lim_{L\to\infty} L^2 \Gamma_v(L) = 0 \ ?$$

Results in two dimensions

- Yes! This can be proved for a potential supported on a strip, i.e., $v(x, y) \geq \gamma > 0$ for $-\delta < x < +\delta$ and $v \equiv 0$ elsewhere.
- Intuitively, such a potential again leads to an effective decomposition of the domain into two congruent parts and hence to an effective degeneracy of the ground state in the infinite-volume limit.

Thank you for your attention!

- J. Kerner and M. Täufer *On the spectral gap of one-dimensional Schrödinger operators on large intervals*, arXiv:2012.09060.
- J. Kerner and M. Täufer *On the spectral gap of higher-dimensional Schrödinger operators on large domains*, arXiv:2110.15110, prov. accept. Asymptotic Analysis
- J. Kerner *A lower bound on the spectral gap of Schrödinger operators with weak potentials of compact support*, arXiv:2103.03813 (might be combined with paper below)
- J. Kerner *A lower bound on the spectral gap of one-dimensional Schrödinger operator*, arXiv:2102.03816, prov. accept. Archiv der Mathematik