On the Korteweg-de Vries limit for the Fermi-Pasta-Ulam system

Younghun Hong

joint with Chulkwang Kwak (Ewha Womans Univ)
Changhun Yang (Chungbuk Nat'l Univ)

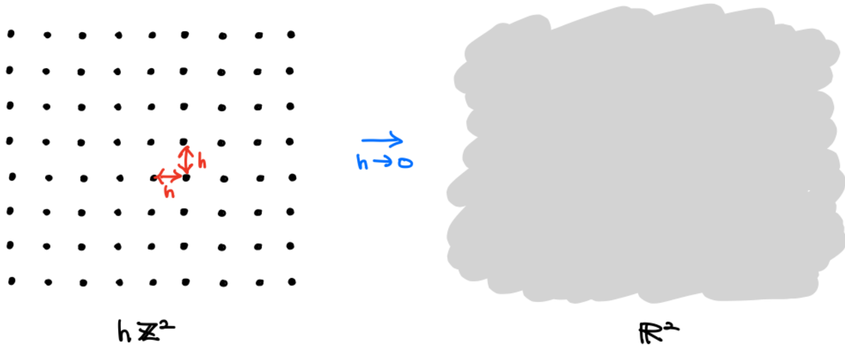
Chung-Ang University

June 28, 2021

Brief introduction on continuum limits

∘ Domain $h\mathbb{Z}^d = \{hm : m \in \mathbb{Z}^d\}$.

∘ Functions $u_h : h\mathbb{Z}^d \to (\mathbb{R} \text{ or } \mathbb{C})$.



∘ (Right) difference operator $\nabla_h^+ = (\nabla_{h,1}^+, \cdots, \nabla_{h,1}^+)$

$$(\nabla_{h,j}^+ u_h)(x) := \frac{u_h(x+he_j) - u_h(x)}{h}, \quad x \in h\mathbb{Z}^d$$

∘ Discrete Laplacian $\Delta_h = \nabla_h \cdot \nabla_h^*$

$$(\Delta_h u_h)(x) := \sum_{j=1}^d \frac{u_h(x+he_j) + u_h(x-he_j) - 2u_h(x)}{h^2}$$

Difference equations (ODEs) on the lattice $h\mathbb{Z}^d$

$$\partial_t u_h = \Delta_h u_h \quad \text{(heat)}; \quad i\partial_t u_h = -\Delta_h u_h \quad \text{(Schrödinger)}; \quad \partial_t^2 u_h = \Delta_h u_h \quad \text{(wave)}$$

$+$ nonlinear variations...

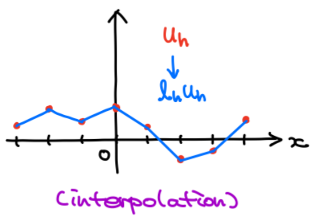
Question: Can we show convergence from a solution $u_h(t)$ to a difference equation to a solution $u(t)$ to the corresponding PDE on $\mathbb{R}^d$ as $h \to 0$?

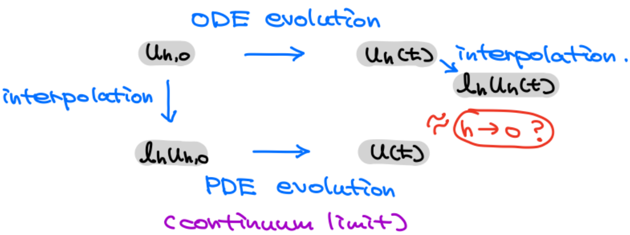
∘ (Linear interpolation) For a function u_h on a lattice $h\mathbb{Z}^d$, we define

$$(l_h u_h)(x) := u_h(hm) + (\nabla_h u_h) \cdot (x - hm) \quad \forall x \in hm + [0,h)^d.$$

Then,

$$\boxed{(l_h u_h)(t,x) \to u(t,x) \quad \text{as } h \to 0?} \qquad \text{(continuum limit)}$$





Motivations

(1) numerical analysis

- FDM (finite difference method)
- finite lattice, time discretization...

(2) solid state physics

- hoping electrons on a crystal
- small amplitude waves $\underset{\text{scaling}}{\leftrightarrow}$ continuum limit setting.
- more geometric lattices: triangular, honeycomb ...

(3) a few others...

- KdV limit from FPUT (in this talk)

Brief introduction on continuum limits

References (for a class of dispersive PDEs)

(1) Continuum limit of NLS[1] on $h\mathbb{Z}^d$

- Ignat-Zuazua 2005, 2009, 2012
- H'-Yang 2019

(2) Continuum limit of fractional NLS on $h\mathbb{Z}^d$

- Kirkpatrick-Lenzmann-Staffilani 2013: general long range interactions
- H'-Yang 2019
- Grande 2019 (arXiv): fractional temporal differentiation is included.

(3) Continuum limit of NLS on a finite lattice

- H'-Kwak-Nakamura-Yang 2021: 2d periodic lattice
- H'-Kwak-Yang 2021 (arXiv): 3d bounded lattice with zero boundary condition

(4) Derivation of nonlinear Dirac equation from NLS on a hexagonal lattice

- Ablowitz-Nixon-Zhu 2009, Ablowitz-Zhu 2012, Arbunich-Sparber 2018.

[1] nonlinear Schrödinger equation

A key aspect for continuum limit of nonlinear dispersive PDEs.

• better estimates for linear flows ⇒ better estimates for nonlinear solutions ⇒ continuum limit

Ex (NLS)

$$\begin{aligned}(l_h u_h)(t) - u(t) &= l_h \left\{ e^{it\Delta_h} u_{h,0} - i \int_0^t e^{i(t-s)\Delta_h} (|u_h|^2 u_h)(s) ds \right\} \\ &\quad - \left\{ e^{it\Delta}(l_h u_{h,0}) - i \int_0^t e^{i(t-s)\Delta} (|u|^2 u)(s) ds \right\} \\ &= (\text{small error})^2 - i \int_0^t e^{i(t-s)\Delta} (|l_h u_h|^2 l_h u_h - |u|^2 u)(s) ds.\end{aligned}$$

If $\|u_h\|_{L_t^p([0,T];L_x^\infty)}$ and $\|u\|_{L_t^p([0,T];L_x^\infty)}$ are bounded, Grönwall's inequality yields continuum limit (+rate of convergence).

[2] by commutator estimates

A tool for $h\mathbb{Z}^d$

Definition (Discrete Fourier transform)

For $u : h\mathbb{Z}^d \to \mathbb{C}$, we define

$$(\mathcal{F}_h u)(\xi) = h^d \sum_{x \in h\mathbb{Z}^d} u(x) e^{-ix\cdot\xi}, \quad \xi \in [-\tfrac{\pi}{h}, \tfrac{\pi}{h})^d.$$

For $g : [-\frac{\pi}{h}, \frac{\pi}{h})^d \to \mathbb{C}$, we define

$$(\mathcal{F}_h^{-1} g)(x) = \frac{1}{(2\pi)^d} \int_{[-\frac{\pi}{h}, \frac{\pi}{h})^d} g(\xi) e^{ix\cdot\xi} d\xi, \quad x \in h\mathbb{Z}.$$

- functions on a lattice $h\mathbb{Z}^d$ $\leftrightarrow$ functions on a periodic box $[-\frac{\pi}{h}, \frac{\pi}{h})^d$.
- Inverse Fourier transform on $h\mathbb{Z}^d$ = Fourier series.
- As $h \to 0$, $\mathcal{F}_h$ and $\mathcal{F}_h^{-1}$ formally converge to the Fourier and the inverse Fourier transform on $\mathbb{R}^d$.

Different dispersion $\Rightarrow$ different linear estimates

Ex) 1d linear Schrödinger equation

$$\Rightarrow \mathcal{F}_h(-\Delta_h u)(\xi) = \mathcal{F}_h\left\{-\frac{u(\cdot+h)+u(\cdot-h)-2u}{h^2}\right\}(\xi)$$
$$= \frac{2-e^{ih\xi}-e^{-ih\xi}}{h^2}(\mathcal{F}_h u)(\xi) = \frac{2-2\cos h\xi}{h^2}(\mathcal{F}_h u)(\xi).$$
$$\Rightarrow (e^{it\Delta_h}u_{h,0})(x) = \frac{1}{2\pi}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{-\frac{2it}{h^2}(1-\cos(h\xi))}(\mathcal{F}_h u_{h,0})(\xi)d\xi.$$

Since $(1-\cos(h\xi))'' = h^2\cos(h\xi)$ $(= 0 \text{ if } \xi = \pm\frac{\pi}{2h})$ and $(1-\cos(h\xi))''' = -h^3\sin(h\xi)$, the standard van der Corput lemma yields a weaker dispersion

$$|e^{it\Delta_h}u_{h,0}(x)| \lesssim \frac{1}{|th|^{1/3}} \quad \text{in } h\mathbb{Z} \quad \left(\leftrightarrow \quad |e^{it\Delta}u_0(x)| \lesssim \frac{1}{|t|^{1/2}} \quad \text{in } \mathbb{R}\right).$$

Brief introduction on continuum limit

Different dispersion $\Rightarrow$ different linear estimates

• A key in our analysis is to find a suitable linear estimates on a discrete setting.
$\Rightarrow$ It generates an interesting set of problems in a harmonic analysis perspective.

Oscillatory integral with a degenerate phase

Ex (Borovyk-Goldberg 2017)

$$\text{Klein-Gordon} \quad \iint_{[-\pi,\pi)^2} e^{\pm it\sqrt{1+2(1-\cos\xi_1)+2(1-\cos\xi_2)}}d\xi_1 d\xi_2.$$

Deep theory in harmonic analysis

- Varchenko 1976, decay $\leftarrow$ Newton's polyhedron.
- Karpushkin 1986, stability.
- Series of works of Phong and Stein....

Summary

• Continuum limit problems are introduced.

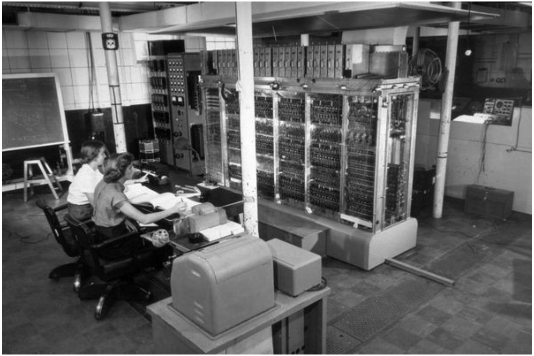
• A core in analysis would be to obtain suitable linear estimates.

• For dispersive equations, we may have different estimates. $\Rightarrow$ oscillatory integral theory is needed.

MANIAC

<u>**MANIAC**</u> (**M**athematical **A**nalyzer **N**umerical **I**ntegrator **A**nd **C**omputer)

∘ an early computer built in Los Alamos National Laboratory (1952-58).

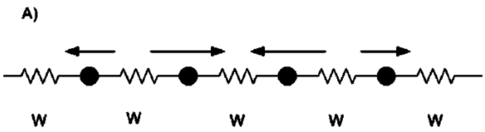


(https://www.computerhistory.org/revolution/supercomputers/10/28/46)

FPUT (Fermi-Pasta-Ulam-Tsingou) system[3]

Physicists Fermi, Pasta, Ulam and Tsingou introduced a simple nonlinear system for numerical simulations (1955).

◦ 1D chain of vibrating strings interacting only with nearest neighbors.



[3] Historically Mary Tsingou's contributions were ignored, and it is formerly called the FPU system.

FPUT system

- There are various settings. In this talk, we consider an infinite chain[4].

$x \in \mathbb{Z}$: the label of strings

$(q(t,x), p(t,x))$: position and momentum of the x-th string at time t.

$V(q) = \frac{q^2}{2} - \frac{q^3}{6}$: standard FPU potential.

$$\begin{cases} \partial_t q(t,x) = p(t,x), \\ \partial_t p(t,x) = V'\big(q(t,x+1) - q(t,x)\big) - V'\big(q(t,x) - q(t,x-1)\big) \end{cases}$$

$$\text{Force} = \underbrace{V'\big(q(t,x+1) - q(t,x)\big) - V'\big(q(t,x) - q(t,x-1)\big)}_{\text{acting on the } x\text{-th string. It depends only on their nearest neighbors.}}$$

[4] finite chain with zero boundary or periodic chain can be considered

FPUT paradox

They anticipated thermalization.

$$\text{single mode} \underset{\text{chaotic interactions}}{\Rightarrow} \text{equally distributed modes}$$

Surprisingly at that time, numerical simulations showed the opposite behavior.

$$\boxed{\text{quasi-periodic motions}}$$

FPUT paradox

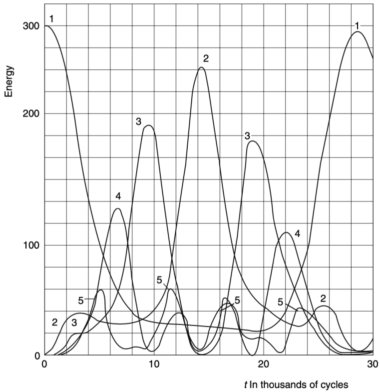


Fig. 4.1. The time evolution of the harmonic energies. The figure is a reproduction of the first one of the original FPU report. Here, $N = 32$ (with $\alpha = 1/4$, $\beta = 0$), and the energy was given initially just to the lowest frequency mode. One sees that the energy, instead of flowing to all the 32 modes, remains confined within a packet of low-frequency modes, namely modes 1 up to 5

[Gallavotti, The Fermi-Pasta-Ulam Problem, p 157]

This phenomenon is known as the FPUT paradox "at that time."

Emergence of KdV

In 1965, physicists Zabusky and Kruskal solved this puzzle discovering the connection to the Kortweg de Vries (KdV) equation.

Zabusky and Kruskal's discovery

Consider the relative displacement $\tilde{q}(t,x) := q(t,x+1) - q(t,x)$.

$$\partial_t^2 \tilde{q}(t,x) = V'\big(\tilde{q}(t,x+1)\big) + V'\big(\tilde{q}(t,x-1)\big) - 2V'\big(\tilde{q}(t,x)\big) \quad \text{(FPUT)}$$

$$\underbrace{h^2 w_+\big(h^3 t, h(x-t)\big) + h^2 w_-\big(h^3 t, h(x+t)\big)}_{\text{counter-propagating KdV flows}} \quad (0 < h \ll 1)$$

solves the equation up to small error in terms of h, where w_+ and w_- solves

$$\partial_t w_\pm \pm \frac{1}{24}\partial_x^3 w_\pm \mp \frac{1}{4}(w_\pm^2)_x = 0 \quad \text{(KdV)}$$

Thus,

$$\boxed{\text{quasi-periodicity of KdV} \quad \Rightarrow \quad \text{quasi-periodicity of FPUT}}$$

Huge literature on the FPUT system

◦ dynamical system.

◦ integrable system.

...

Rigorous proof (by a dynamical system approach)

Theorem (Schneider-Wayne '00)

Suppose that for $j=1,2$,

$$\|\psi_j\|_{H^{14+}(\mathbb{R})}<\infty,\quad \|\langle x\rangle^7\psi_j\|_{H^7(\mathbb{R})}<\infty,$$

$$\left\|\langle x\rangle^2\left(\tilde{q}_0(x)-h^2\psi_1(hx)\right)\right\|_{\ell^2(\mathbb{Z})}\leq Ch^4,\quad \left\|\langle x\rangle^2\left(\tilde{q}_1(x)-h^3\psi_2(hx)\right)\right\|_{\ell^2(\mathbb{Z})}\leq Ch^5$$

$$\sum_x \tilde{q}_1(x)=0,\quad \int_{\mathbb{R}}\psi_2(x)dx=0.$$

Let $w_\pm(t,x)$ be solutions to KdVs with initial data $w_{\pm,0}(x)=\frac{1}{2}(\psi_1(x)\pm\int_{-\infty}^x\psi_2 dy)$. Then, for sufficently small $h>0$, a solution $\tilde{q}(t,x)$ to FPU with initial data $(\tilde{q}_0,\tilde{q}_1)$ satisfies

$$\sup_{t\in[0,\frac{1}{h^2}]}\left\|\tilde{q}(t,x)-h^2w_+(h^3t,h(x-t))-h^2w_-(h^3t,h(x+t))\right\|_{\ell^\infty(\mathbb{Z})}\lesssim h^{\frac{7}{2}}.$$

Summary

• Fermi, Pasta, Ulam and Tsingou observed quasi-periodic motions of their numerical model (1955).

• Zabusky and Kruskal discovered the connection between FPUT and KdV (1965). It explains quasi-periodicity of the FPUT system.

• Mathematically, Schneider and Wayne proved the connection using a dynamical system approach (2000).

Reformulation of the problem

• We revisit the problem using the discrete Fourier transform and the dispersive PDE approach.

$\Rightarrow$ Can we relax some assumptions in the previous result?

Reformulation of the problem

Recall

$$\boxed{\partial_t^2 \tilde{q}(t,x) = V'\big(\tilde{q}(t,x+1)\big) + V'\big(\tilde{q}(t,x-1)\big) - 2V'\big(\tilde{q}(t,x)\big)}$$

$$\tilde{q}(t,x) \approx h^2 w_+(h^3 t, h(x-t)) + h^2 w_-(h^3 t, h(x+t)) : I(\subset \mathbb{R}) \times \mathbb{Z} \to \mathbb{R}$$

Scaling

Introduce $\tilde{q}_h(t,x) := \frac{1}{h^2}\tilde{q}(\frac{t}{h^3}, \frac{x}{h}) : \mathbb{R} \times h\mathbb{Z} \to \mathbb{R}$.

$$\boxed{\Rightarrow h^6 \partial_t^2 \tilde{q}_h(t,x) = \Delta_h\big(V'\big(h^2 \tilde{q}_h(t,x)\big)\big)}$$

where

$$(\Delta_h u)(x) = \frac{u(x+h) + u(x-h) - 2u(x)}{h^2}, \quad \forall x \in h\mathbb{Z}$$

is a discrete Laplacian on $h\mathbb{Z}$.

Reformulation of the problem

In particular, when $V(r)=\frac{r^2}{2}+\frac{r^3}{6}$ (standard FPU potential),

$$V'(r)=r+\frac{r^2}{2}=\text{linear}+\text{quadratic}.$$

FPU is a discrete nonlinear wave equation

$$\begin{cases}\partial_t^2\tilde{q}_h-\frac{1}{h^4}\Delta_h\tilde{q}_h=\frac{1}{2h^2}\Delta_h(\tilde{q}_h^2),\\ \tilde{q}_h(0)=\tilde{q}_{h,0},\\ \partial_t\tilde{q}_h(0)=\tilde{q}_{h,1}.\end{cases}$$

By Duhamel's formula, we write

$$\tilde{q}_h(t)=\cos\left(\frac{t\sqrt{-\Delta_h}}{h^2}\right)\tilde{q}_{h,0}+\sin\left(\frac{t\sqrt{-\Delta_h}}{h^2}\right)\frac{h^2}{\sqrt{-\Delta_h}}\tilde{q}_{h,1}$$
$$-\frac{1}{2}\int_0^t\sin\left(\frac{(t-s)\sqrt{-\Delta_h}}{h^2}\right)\sqrt{-\Delta_h}\Big\{\tilde{q}_h(s)\Big\}^2ds.$$

- On Fourier side,

$$\cos\left(\frac{t\sqrt{-\Delta_h}}{h^2}\right) \leftrightarrow \cos\left(\frac{2t}{h^3}\left|\sin\frac{h\xi}{2}\right|\right), \quad \sin\left(\frac{t\sqrt{-\Delta_h}}{h^2}\right) \leftrightarrow \sin\left(\frac{2t}{h^3}\left|\sin\frac{h\xi}{2}\right|\right)$$

Direct calculations (on Fourier side) yield

$$\begin{aligned}
\cos\left(\frac{t\sqrt{-\Delta_h}}{h^2}\right) &= \frac{1}{2}e^{\frac{t}{h^2}\nabla_h} + \frac{1}{2}e^{-\frac{t}{4h^2}\nabla_h} \\
\sin\left(\frac{\sqrt{-\Delta_h}}{h^2}\right)\frac{h^2}{\sqrt{-\Delta_h}} &= \frac{h^2}{2}e^{\frac{t}{h^2}\nabla_h}\nabla_h^{-1} - \frac{h^2}{2}e^{-\frac{t}{4h^2}\nabla_h}\nabla_h^{-1}, \\
\sin\left(\frac{(t-s)\sqrt{-\Delta_h}}{h^2}\right)\sqrt{-\Delta_h} &= \frac{1}{2}e^{\frac{t-s}{h^2}\nabla_h}\nabla_h - \frac{1}{2}e^{-\frac{t-s}{h^2}\nabla_h}\nabla_h,
\end{aligned}$$

where

$$\nabla_h \quad \leftrightarrow \quad \tfrac{2i}{h}\sin(\tfrac{h\xi}{2}).$$

Reformulation of the problem

Separate the operators $e^{\mp \frac{t}{h^2}\nabla_h}$.

$\Rightarrow$ a system of "two coupled" equations for $\tilde{q}_h^+$ and $\tilde{q}_h^-$,

$$\boxed{\tilde{q}_h^\pm(t) = \frac{1}{2} e^{\mp \frac{t}{h^2}\nabla_h} \tilde{q}_{h,0}^\pm \mp \frac{1}{4}\int_0^t e^{\mp \frac{(t-s)}{h^2}\nabla_h} \nabla_h \Big\{ \tilde{q}_h^+(s) + \tilde{q}_h^-(s) \Big\}^2 ds}$$

with initial data

$$\tilde{q}_{h,0}^\pm = \frac{1}{2}\Big\{ \tilde{q}_{h,0} \mp h^2 \nabla_h^{-1} \tilde{q}_{h,1} \Big\}$$

and

$$\tilde{q}_h(t,x) = \tilde{q}_h^+(t,x) + \tilde{q}_h^-(t,x).$$

Reformulation of the problem

Next, we introduce

$$u_h^{\pm}(t) := e^{\pm \frac{t}{h^2}\partial_h} \tilde{q}_h^{\pm}(t),$$

where ∂_h is the Fourier multiplier of symbol $i\xi$.

- $e^{\pm \frac{t}{h^2}\partial_h}$ is an almost translation. If $t = h^3 k$ with $k \in \mathbb{Z}$,

$$e^{\pm \frac{t}{h^2}\partial_h} u(t,x) = e^{\pm hk\partial_h} u(t,x) = u_h^{\mp}(t, x \pm hk) = u(t, x \pm \tfrac{t}{h^2}).$$

- Recall the Zabusky and Kruskal ansatz

$$\tilde{q}(t,x) = h^2 w_+(h^3 t, h(x-t)) + h^2 w_-(h^3 t, h(x+t)).$$

After scaling, it makes sense to translate $\tilde{q}_h^{\pm}(t)$ by $\pm\frac{t}{h^2}$.

Finally, ...

$\Rightarrow$ **coupled FPU**

$$u_h^{\pm}(t)=S_h^{\pm}(t)u_{h,0}^{\pm}\mp\frac{1}{4}\int_0^t S_h^{\pm}(t-s)\nabla_h\Big\{u_h^{\pm}(s)+e^{\pm\frac{2s}{h^2}\partial_h}u_h^{\mp}(s)\Big\}^2ds,$$

with initial data

$$u_{h,0}^{\pm}=\frac{1}{2}\Big\{\tilde{q}_{h,0}\mp h^2\partial_h^{-1}\tilde{q}_{h,1}\Big\},$$

where

$$S_h^{\pm}(t)=e^{\mp\frac{t}{h^2}(\nabla_h-\partial_h)}\quad\leftrightarrow\quad e^{\pm\frac{it}{h^2}(\xi+\frac{2}{h}\sin(\frac{h\xi}{2}))}.$$

• The coupled FPU is nothing but a reformulation of FPU following Zabusky and Kruskal's idea.

• However, it is easier to understand what is happening in a Fourier analysis perspective.

Reformulation of the problem: From coupled to decoupled FPU

Observation 1

If $u_h^{\pm}(t)$ behaves linearly (this must be true locally in time),

$$\begin{aligned} e^{\pm\frac{2s}{h^2}\partial_h}u_h^{\mp}(s,x) &\approx e^{\pm\frac{2s}{h^2}\partial_h}S_h^{\mp}(s)u_0^{\mp}(x) \\ &= \frac{1}{2\pi}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{\pm\frac{2is}{h}\xi}e^{\mp\frac{is}{h^2}(\xi+\frac{2}{h}\sin(\frac{h\xi}{2}))}(\mathcal{F}_h u_0^{\mp})(\xi)e^{ix\xi}d\xi \\ &= \frac{1}{2\pi}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{\pm\frac{is}{h^2}(\xi-\frac{2}{h}\sin(\frac{h\xi}{2}))+ix\xi}(\mathcal{F}_h u_0^{\mp})(\xi)d\xi. \end{aligned}$$

Its group velocity diverges

$$\mp\frac{1}{h^2}(1+\cos(\tfrac{h\xi}{2}))\to\mp\infty,\quad \xi\in[-\tfrac{\pi}{h},\tfrac{\pi}{h}).$$

$\rightsquigarrow$ **Fast dispersion!**

Remove the cross terms $\Rightarrow$ **decoupled FPU**

$$v_h^{\pm}(t)=S_h^{\pm}(t)u_{h,0}^{\pm}\mp\frac{1}{4}\int_0^t S_h^{\pm}(t-s)\nabla_h\left\{v_h^{\pm}(s)\right\}^2ds,$$

Observation 2

The linear propagator $S_h^{\pm}(t)$ formally converges to the Airy flow $S^{\pm}(t)$, because by Taylor's theorem,

$$\mp\tfrac{1}{h^2}\left(\tfrac{2}{h}\sin\left(\tfrac{h\xi}{2}\right)-\xi\right)=\pm\left(\tfrac{\xi^3}{24}-\tfrac{h^2\xi^5}{1920}+\cdots\right)\to\pm\tfrac{\xi^3}{24}.$$

decoupled FPU $\Rightarrow$ **KdV**

$$w^{\pm}(t)=S^{\pm}(t)u_0^{\pm}\mp\frac{1}{4}\int_0^t S^{\pm}(t-t_1)\partial_x\Big\{w^{\pm}(t_1)\Big\}^2dt_1$$

where

$$S^{\pm}(t)=e^{\mp\frac{t}{24}\partial_x^3}.$$

Summary

3 equations

(1) coupled FPU (=FPU)

$$u_h^{\pm}(t) = S_h^{\pm}(t)u_{h,0}^{\pm} \mp \frac{1}{4}\int_0^t S_h^{\pm}(t-s)\nabla_h\Big\{u_h^{\pm}(s) + e^{\pm\frac{2s}{h^2}\partial_h}u_h^{\mp}(s)\Big\}^2 ds$$

- We expect that the coupled terms $e^{\pm\frac{2s}{h^2}\partial_h}u_h^{\mp}(s)$ will disappear as $h \to 0$.

(2) decoupled FPU

$$v_h^{\pm}(t) = S_h^{\pm}(t)u_{h,0}^{\pm} \mp \frac{1}{4}\int_0^t S_h^{\pm}(t-s)\nabla_h\Big\{v_h^{\pm}(s)\Big\}^2 ds$$

- We expect that $S_h^{\pm}(t) \to S^{\pm}(t)$ as $h \to 0$.

(3) KdV

$$w^{\pm}(t) = S^{\pm}(t)u_0^{\pm} \mp \frac{1}{4}\int_0^t S^{\pm}(t-s)\partial_x\Big\{w^{\pm}(s)\Big\}^2 ds$$

Main results

Definition (L^p norm)

$$\|u\|_{L^p(h\mathbb{Z})} := \begin{cases} \left\{ h \sum_{x \in h\mathbb{Z}} |u(x)|^p \right\}^{1/p} & \text{if } 1 \leq p < \infty, \\ \sup_{x \in h\mathbb{Z}} |u(x)| & \text{if } p = \infty. \end{cases}$$

Definition (Sobolev norm)

$$\|u\|_{W^{s,p}(h\mathbb{Z})} : = \|(1 - \Delta_h)^{\frac{s}{2}} u\|_{L^p(h\mathbb{Z})},$$
$$\|u\|_{\dot{W}^{s,p}(h\mathbb{Z})} : = \|(-\Delta_h)^{\frac{s}{2}} u\|_{L^p(h\mathbb{Z})}.$$

Main results (1): Convergence to the decoupled FPU

Theorem (H.-Kwak-Yang 2021; from couple to decoupled FPU)

Suppose that

$$\sup_{h\in(0,1]} \|u_{h,0}^{\pm}\|_{H^s(h\mathbb{Z})} < \infty$$

for some $s \in (0,1]$. *Let*

$$\left(u_h^+(t), u_h^-(t)\right) \quad \Big(\text{resp. } \left(v_h^+(t), v_h^-(t)\right)\Big)$$

be the solution to the coupled FPU (resp. the decoupled FPU) with discretized initial data $(u_{h,0}^+, u_{h,0}^-)$. *Then, there exists* $T > 0$, *independent of* $h \in (0,1]$, *such that*

$$\left\|u_h^{\pm}(t) - v_h^{\pm}(t)\right\|_{C_t([-T,T];L_x^2(h\mathbb{Z}))} \lesssim h^s \|u_0^{\pm}\|_{H^s(\mathbb{R})}.$$

Main results: Linear interpolation

Definition (Linear interpolation operator)

For a function $f_h : h\mathbb{Z} \to \mathbb{C}$ on a lattice domain, we define

$$(I_h f_h)(x) := f_h(x_m) + (\partial_h^+ f_h)(x_m) \cdot (x - x_m), \quad \forall x \in x_m + [0, h),$$

Main results (2): Convergence to KdV

Theorem (H.-Kwak-Yang 2021; from decoupled FPU to KdV)

Suppose that

$$\sup_{h\in(0,1]} \|u_{h,0}^{\pm}\|_{H^s(h\mathbb{Z})} < \infty$$

for some $s \in (\frac{3}{4}, 1]$. *Let*

$$\left(v_h^+(t), v_h^-(t)\right) \quad \Big(\text{resp. } \left(w^+(t), w^-(t)\right)\Big)$$

be the solution to the decoupled FPU (resp. the KdVs) with discretized initial data $(u_{h,0}^+, u_{h,0}^-)$ *(resp. with initial data* $(u_0^+, u_0^-) = (l_h u_{h,0}^+, l_h u_{h,0}^-)$*). Then, there exists* $T > 0$*, independent of* $h \in (0, 1]$*, such that*

$$\left\| l_h v_h^{\pm}(t) - w^{\pm}(t) \right\|_{C_t([-T,T];L_x^2(\mathbb{R}))} \lesssim h^{\frac{2s}{5}} \|u_{h,0}^{\pm}\|_{H^s(h\mathbb{Z})}.$$

Combining two theorems, we establish convergence from FPU to KdV as $h \to 0$ **"for $H^{\frac{3}{4}+}$ data"**.

What do we need?

- Since we would like to prove convergence as $h \to 0$, we need

inequalities which hold uniformly in $h \in (0,1]$

- On a lattice $h\mathbb{Z}$, we have the embedding $L^1(h\mathbb{Z}) \hookrightarrow L^\infty(h\mathbb{Z})$, which is not true on $\mathbb{R}$:

$$\|u\|_{L^\infty(h\mathbb{Z})} = \sup_{x\in h\mathbb{Z}} |u(x)| \le \sum_{x\in h\mathbb{Z}} |u(x)| = \frac{1}{h}\|u\|_{L^1(h\mathbb{Z})}.$$

For fixed $h > 0$, we have more inequalities, but we cannot use them if constant blows up as $h \to 0$.

- uniform Sobolev inequalities, uniform Littlewood-Paley inequalties... (H.-Yang '19).

Dispersive PDE techniques

What do we need? (continued)

- To handle low regularity data, we need

various uniform dispersive/smoothing estimates

- Fourier analysis, harmonic analysis, and dispersive PDE techniques must be useful!

Example 1: local smoothing estimate

For the Airy flow

$$S^{\pm}(t) \quad \leftrightarrow \quad e^{\pm i \frac{t}{24}\xi^3},$$

Theorem (Local smoothing estimate for Airy flow; Kenig-Ponce-Vega '91)

$$\|\partial_x S^{\pm}(t)u_0\|_{L_x^\infty(\mathbb{R};L_t^2(\mathbb{R}))} \lesssim \|u_0\|_{L^2(\mathbb{R})}.$$

For the linear FPU flow

$$S_h^{\pm}(t) \quad \leftrightarrow \quad e^{\mp \frac{it}{h^2}(\frac{2}{h}\sin(\frac{h\xi}{2})-\xi)},$$

Theorem (Local smoothing estimate for linear FPU flow; H.-Kwak-Yang)

$$\|\partial_h S_h^{\pm}(t)u_{h,0}\|_{L_x^\infty(h\mathbb{Z};L_t^2(\mathbb{R}))} \lesssim \|u_{h,0}\|_{L^2(h\mathbb{Z})}.$$

Example 1: local smoothing estimate

More generally, we have

Lemma

If $p : \mathbb{R} \to \mathbb{R}$ is a C^1 monotone function, then

$$\left\| e^{itp(-i\nabla)} u_0 \right\|_{L_x^\infty L_t^2} \lesssim \left\| \frac{1}{\sqrt{|p'(-i\nabla)|}} u_0 \right\|_{L^2}.$$

Sketch of the proof

$$\begin{aligned}
\left\| e^{itp(-i\nabla)} \sqrt{|p'(-i\nabla)|} u_0 \right\|_{L_t^2} &= \left\| \int_{\mathbb{R}} e^{itp(\xi)} e^{ix\xi} \sqrt{|p'(\xi)|} \hat{u}_0(\xi) d\xi \right\|_{L_t^2} \\
&= \left\| \int_{\mathbb{R}} e^{it\tau} e^{ix\xi} \frac{1}{\sqrt{|p'(\xi)|}} \hat{u}_0(\xi) d\tau \right\|_{L_t^2} \quad (\text{with } \tau = p(\xi)) \\
&\sim \left\| e^{ix\xi} \frac{1}{\sqrt{|p'(\xi)|}} \hat{u}_0(\xi) \right\|_{L_\tau^2} = \left\{ \int_{\mathbb{R}} \frac{|\hat{u}_0(\xi)|^2}{|p'(\xi)|} d\tau \right\}^{1/2} \\
&= \|\hat{u}_0\|_{L_\xi^2} \sim \|u_0\|_{L^2}.
\end{aligned}$$

Our setting

$$\sqrt{p'(\xi)} = \sqrt{\frac{1}{h^2}\left(1 - \cos\left(\frac{h\xi}{2}\right)\right)}$$

Good algebra by the double angle and the half-angle formulas

$$\frac{\frac{2}{h}|\sin(\frac{h\xi}{2})|}{\sqrt{\frac{1}{h^2}(1-\cos(\frac{h\xi}{2}))}} = \frac{2|2\sin(\frac{h\xi}{4})\cos(\frac{h\xi}{4})|}{\sqrt{2}|\sin(\frac{h\xi}{4})|} = 2\sqrt{2}|\cos(\frac{h\xi}{4})| \lesssim 1!$$

Thus,

$$\|S_h^{\pm}(t)u_{h,0}\|_{L_x^\infty(h\mathbb{Z};L_t^2(\mathbb{R}))} \lesssim \left\|\frac{1}{\sqrt{|p'(-i\nabla)|}}u_0\right\|_{L^2} \lesssim \|\partial_h^{-1}u_{h,0}\|_{L^2(h\mathbb{Z})}.$$

Example 1: local smoothing estimate

In general, we don't expect smoothing in a discrete setting.

Failure of local smoothing for Schrödinger flow

For the linear Schrödinger flow $e^{it\Delta_h}$,

$$p(\xi) = \frac{2(1-\cos(h\xi))}{h^2} \quad \Rightarrow \quad p'(\xi) = \frac{2\sin(h\xi)}{h} \approx 0 \text{ near } \pm\frac{\pi}{h}.$$

No local smoothing for the discrete Schrödinger equation [Ignat-Zuazua 09]

Example 2: $X^{s,b}$ blinear estimates

Definition ($X^{s,b}$ space for KdV)

$$\|u\|_{X_{\pm}^{s,b}} := \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \langle \xi \rangle^{2s} \langle \tau - \xi^3 \rangle^{2b} |\tilde{u}(\tau, \xi)|^2 d\xi d\tau \right\}^{\frac{1}{2}}.$$

Lemma (Kenig-Ponce-Vega '93)

For $s \geq -\frac{3}{4}$, there exist $b = b(s) > \frac{1}{2}$ and $\delta = \delta(b) > 0$ such that

$$\left\| \nabla (uv) \right\|_{X^{s,b-1+\delta}} \lesssim \|u\|_{X^{s,b}} \|v\|_{X^{s,b}}.$$

• The bilinear estimates are a fundamental tool in dispersive PDEs to prove low regularity well-posedness.

Example 2: $X^{s,b}$ blinear estimates

Definition

$$\|u_h\|_{X^{s,b}_{h,\pm}} := \left\{\int_{-\infty}^{\infty}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \langle\xi\rangle^{2s}\langle\tau \mp \tfrac{1}{h^2}(\xi - \tfrac{2}{h}\sin(\tfrac{h\xi}{2}))\rangle^{2b}|\tilde{u}_h(\tau,\xi)|^2 d\xi d\tau\right\}^{\frac{1}{2}}.$$

Lemma (H.-Kwak-Yang)

For $s' \geq s \geq 0$, there exist $b = b(s) > \frac{1}{2}$ and $\delta = \delta(b) > 0$ such that

$$\left\|\nabla_h(u_h^\pm \cdot v_h^\pm)\right\|_{X^{s,b-1+\delta}_{h,\pm}} \lesssim \|u_h^\pm\|_{X^{s,b}_{h,\pm}}\|v_h^\pm\|_{X^{s,b}_{h,\pm}},$$

$$\left\|\nabla_h\left(e^{\pm\frac{2t}{h^2}\partial_h}u_h^\mp \cdot e^{\pm\frac{2t}{h^2}\partial_h}v_h^\mp\right)\right\|_{X^{s,b-1+\delta}_{h,\pm}} \lesssim h^{s'-s}\|u_h^\mp\|_{X^{s',b}_{h,\mp}}\|v_h^\mp\|_{X^{s',b}_{h,\mp}},$$

$$\left\|\nabla_h\left(u_h^\pm \cdot e^{\pm\frac{2t}{h^2}\partial_h}v_h^\mp\right)\right\|_{X^{s,b-1+\delta}_{h,\pm}} \lesssim h^{s'-s}\|u_h^\pm\|_{X^{s',b}_{h,\pm}}\|v_h^\mp\|_{X^{s',b}_{h,\mp}}.$$

- The proof again heavily uses the good algebras.

And more...

In the joint work with Chulkwang Kwak and Changhun Yang, we proved:

- maximal function estimates for the linear FPU flows,
- Strichartz estimates for the linear FPU flows,
- how the linear interpolation works in this setting.

Collecting all, we proved convergence from FPU to KdV.

Thank you for your attention!