Incompressible viscous fluids in the plane and SPDEs on graphs

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Asia-Pacific Analysis and PDE Seminar

November 21/22, 2021

The equation

We consider here

some particles moving together with an incompressible flow in $\mathbb{R}^2$, with stream function H.

If $u(t,x)$ is the density of the particles at time $t \geq 0$ and position $x \in \mathbb{R}^2$, then the function $u(t,x)$ satisfies the Liouville equation

$$\begin{cases} \partial_t u(t,x) = \left\langle \bar{\nabla} H(x), \nabla u(t,x) \right\rangle, & t > 0, \quad x \in \mathbb{R}^2, \\ u(0,x) = \varphi(x), & x \in \mathbb{R}^2. \end{cases} \tag{1}$$

Suppose now that the flow has a small viscosity and the particles take part in a slow reaction, with a deterministic and a stochastic component, as described by the equation

$$\begin{cases} \partial_t \hat{u}_\epsilon(t,x) = & \dfrac{\epsilon}{2}\,\Delta \hat{u}_\epsilon(t,x) + \left\langle \bar{\nabla} H(x), \nabla \hat{u}_\epsilon(t,x)\right\rangle \\ & \\ & +\epsilon\, b(\hat{u}_\epsilon(t,x)) + \sqrt{\epsilon}\, g(\hat{u}_\epsilon(t,x))\partial_t \mathcal{W}(t,x), \\ & \\ \hat{u}_\epsilon(0,x) = & \varphi(x), \quad x \in \mathbb{R}^2. \end{cases} \tag{2}$$

Here, $0 < \epsilon << 1$ is a small parameter, included in equation (2) in such a way that all perturbation terms have strength of the same order, as $\epsilon \downarrow 0$.

The problem

It is not difficult to check that for every fixed $T > 0$ and $\eta > 0$

$$\lim_{\epsilon \to 0} \mathbb{P}\left(\sup_{t \in [0,T]} |\hat{u}_\epsilon(t,x) - u(t,x)| > \eta \right) = 0,$$

uniformly with respect to x in a bounded domain of $\mathbb{R}^2$.

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But

on large time intervals of order ϵ^{-1}, there is a non-trivial limit and the difference $\hat{u}_\epsilon(t,x) - u(t,x)$ can have order 1, as $\epsilon \downarrow 0$.

To describe the long-time behavior of the system it is convenient to define

$$u_\epsilon(t,x) := \hat{u}_\epsilon(t/\epsilon, x), \qquad t \geq 0, \quad x \in \mathbb{R}^2.$$

With this change of time, the new function $u_\epsilon(t,x)$ solves the equation

$$\begin{cases} \partial_t u_\epsilon(t,x) = & \dfrac{1}{2}\,\Delta u_\epsilon(t,x) + \dfrac{1}{\epsilon}\left\langle \bar{\nabla} H(x), \nabla u_\epsilon(t,x)\right\rangle \\ & + b(u_\epsilon(t,x)) + g(u_\epsilon(t,x))\partial_t \mathcal{W}(t,x), \\ u_\epsilon(0,x) = & \varphi(x), \quad x \in \mathbb{R}^2, \end{cases} \tag{3}$$

for some spatially homogeneous Wiener process $\mathcal{W}(t,x)$.

Our goal

Here, we are interested in

the limiting behavior of the solution $u_\epsilon(t, x)$ of equation (3), as $\epsilon \downarrow 0$, in any finite time interval.

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In particular, we will see that, in order to describe the limit of $u_\epsilon(t, x)$,

one should consider SPDEs on a non standard setting, where the space variable changes on the graph Γ obtained by identifying all points in each connected component of the level sets of the Hamiltonian H.

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- There exists a constant $c>0$ such that

$$H(x)\geq c\,|x|^2,\ \ |\nabla H(x)|\geq c\,|x|,\ \ \Delta H(x)\geq c,$$

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For convenience, we assume

$$\min_{x\in\mathbb{R}^2} H(x)=0.$$

The noise

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This means that $\mathcal{W}$ is a Gaussian random field on some $(\mathbf{\Omega}, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P})$, such that

- the mapping $(t,x) \in [0+\infty) \times \mathbb{R}^2 \mapsto \mathcal{W}(t,x)$ is continuous in $t \geq 0$ and measurable in both variables, **P**-almost surely;
- for each $x \in \mathbb{R}^2$, the process $\mathcal{W}(t,x)$, $t \geq 0$, is a one-dimensional Wiener process;
- for every $t, s \geq 0$ and $x, y \in \mathbb{R}^2$

$$\mathbf{E}\, \mathcal{W}(t,x)\mathcal{W}(s,y) = (t \wedge s)\, \Lambda(x-y),$$

  where Λ is the Fourier transform of the spectral measure μ.

Notice that the spatially homogeneous Wiener processes can be represented as

$$\mathcal{W}(t,x) = \sum_{j=1}^{\infty} \widehat{u_j m}(x)\beta_j(t),$$

where $\{u_j\}$ is an orthonormal basis of $L^2_{(s)}(\mathbb{R}^2, \mu)$ and $\{\beta_j\}$ is a sequence of independent Brownian motions.

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In what follows, we assume that

$$\mu(dx) = m(x)\,dx,$$

for some $m \in L^p(\mathbb{R}^2)$, with $p > 1$, and we will distinguish the case $p = 1$ and the case $p > 1$.

The coefficients

We assume that

$$b, g : \mathbb{R}^2 \to \mathbb{R} \text{ are Lipschitz continuous.}$$

[1] It is the dual of the closure of $\mathcal{S}(\mathbb{R}^2)$ w.r.t. the scalar product $\langle \hat{\mu}, \varphi \star \psi_{(s)} \rangle$.

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For every $\rho \in L^1(\mathbb{R}^2)$, $u \in L^2(\mathbb{R}^2, \rho\, dx)$ and v in the reproducing kernel[1] RK of $\mathcal{W}$, we define

$$B(u)(x) = b(u(x)), \quad [G(u)v](x) = g(u(x))v(x), \quad x \in \mathbb{R}^2.$$

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It follows

- $B : L^2(\mathbb{R}^2, \rho\, dx) \to L^2(\mathbb{R}^2, \rho\, dx)$ is Lipschitz continuous,
- $G : L^2(\mathbb{R}^2, \rho\, dx) \to \mathcal{L}_2(RK, L^2(\mathbb{R}^2, \rho\, dx))$ is Lipschitz continuous, when $p = 1$.

[1] It is the dual of the closure of $\mathcal{S}(\mathbb{R}^2)$ w.r.t. the scalar product $\langle \hat{\mu}, \varphi \star \psi_{(s)} \rangle$.

Under the above conditions and suitable other conditions on ρ,

for any $T > 0$ and $q \geq 1$, equation (3) admits a unique mild solution $u_\epsilon \in L^q(\mathbf{\Omega}; C([0, T]; L^2(\mathbb{R}^2, \rho\, dx))$.

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This means that there exists a unique adapted process $u_\epsilon \in L^q(\boldsymbol{\Omega}; C([0,T]; L^2(\mathbb{R}^2, \rho\, dx))$, such that

$$u_\epsilon(t) = S_\epsilon(t)\varphi + \epsilon \int_0^t S_\epsilon(t-s) B(u_\epsilon(s))\, ds$$

$$+\sqrt{\epsilon} \int_0^t S_\epsilon(t-s)\, G(u_\epsilon(s))\, d\mathcal{W}(s),$$

where $S_\epsilon(t)$ is the semigroup associated with the operator

$$\hat{\mathcal{L}}_\epsilon \varphi(x) = \frac{1}{2}\Delta\varphi(x) + \frac{1}{\epsilon}\left\langle \bar{\nabla} H(x), \nabla\varphi(x) \right\rangle, \qquad x \in \mathbb{R}^2.$$

The linear deterministic problem

For every $\epsilon > 0$, we consider the Cauchy problem

$$\begin{cases} \partial_t v_\epsilon(t,x) = \mathcal{L}_\epsilon v_\epsilon(t,x), & t > 0, \quad x \in \mathbb{R}^2, \\ \\ v_\epsilon(0,x) = \varphi(x), & x \in \mathbb{R}^2, \end{cases}$$

where, we recall, $\mathcal{L}_\epsilon$ is the second order uniformly elliptic differential operator defined by

$$\mathcal{L}_\epsilon v(x) = \frac{1}{2}\Delta v(x) + \frac{1}{\epsilon}\left\langle \bar{\nabla} H(x), \nabla v(x) \right\rangle, \qquad x \in \mathbb{R}^2.$$

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Namely,

$$v_\epsilon(t,x) = S_\epsilon(t)\varphi(x) = \mathbb{E}_x\,\varphi(X_\epsilon(t)), \qquad x \in \mathbb{R}^2,$$

where $X_\epsilon(t)$ is the solution of the SDE

$$dX_\epsilon(t) = \frac{1}{\epsilon}\,\bar{\nabla} H(X_\epsilon(t))\,dt + dw(t),$$

for some 2-dimensional Brownian motion $w(t)$, defined on a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$.

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Clearly, the first fundamental goal is studying the limiting behavior of the semigroup $S_\epsilon(t)$, as $\epsilon \downarrow 0$.

Some notations

For every $z \geq 0$, we denote by $C(z)$ the z-level set

$$C(z) = \left\{x \in \mathbb{R}^2 \,:\, H(x) = z\right\} = \bigcup_{k=1}^{N(z)} C_k(z).$$

If $X(t)$ is the solution of the Hamiltonian system

$$\dot{X}(t) = \bar{\nabla} H(X(t)),$$

for every $x \in \mathbb{R}^2$ we have

$$X(0) = x \Longrightarrow X(t) \in C_{k(x)}(H(x)), \qquad t \geq 0,$$

where $C_{k(x)}(H(x))$ is the connected component of the level set $C(H(x))$, containing x.

Now, for every $z \geq 0$ and $k = 1, \ldots, N(z)$, we define

$$T_k(z) = \oint_{C_k(z)} \frac{1}{|\nabla H(x)|} \, dl_{z,k},$$

where $dl_{z,k}$ is the length element on $C_k(z)$.

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Moreover, the probability measure

$$d\mu_{z,k} := \frac{1}{T_k(z)} \frac{1}{|\nabla H(x)|} \, dl_{z,k}$$

is invariant for the Hamiltonian system on the level set $C_k(z)$.

The graph Γ

If we identify all points in $\mathbb{R}^2$ belonging to the same connected component of a given level set $C(z)$ of the Hamiltonian H,

we obtain a graph Γ, given by several intervals $I_1, \dots I_n$ and vertices $O_1, \dots, O_m$.

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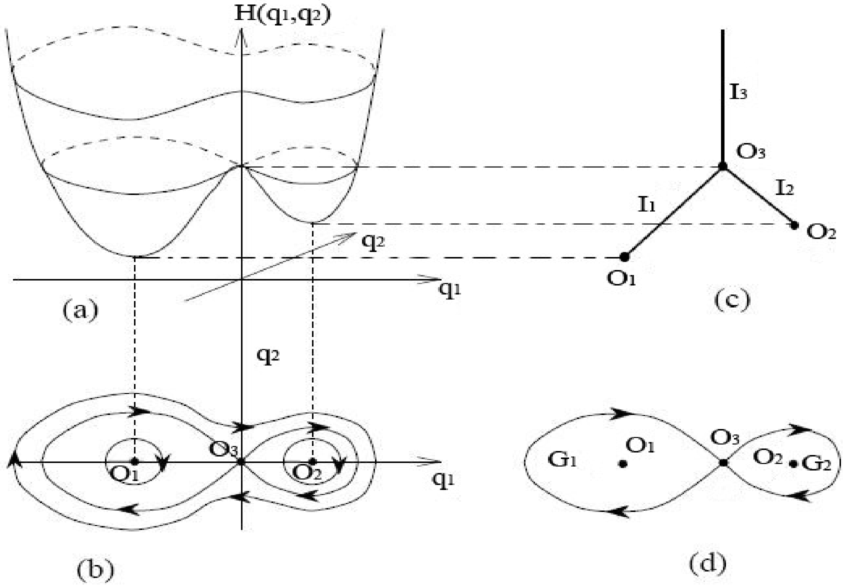
The vertices will be of two different types,

external and internal vertices.

External vertices correspond to local extrema of H, while internal vertices correspond to saddle points of H.


H(q1,q2)
I3
O3
I2
I1
q2
O2
O1
q1
(a)
(c)
q2
O3
O1
O2
q1
G1
O1
O3
O2
G2
(b)
(d)

The identification map

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We have

$$\Pi(x) = (H(x), k(x)),$$

where $k(x)$ denotes the number of the interval on the graph Γ, containing the point $\Pi(x)$.

Both $k(x)$ and $H(x)$ are first integrals for the Hamiltonian system

$$\dot{X}(t) = \bar{\nabla} H(X(t)).$$

A limiting result

Freidlin and Wentcell in 2002 have studied

the limiting behavior, as $\epsilon \downarrow 0$, of the (non Markov) process $\Pi_\epsilon(t) := \Pi(X_\epsilon(t))$, $t \geq 0$, in the space $C([0, T]; \Gamma)$, for any fixed $T > 0$ and $x \in \mathbb{R}^2$.

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Namely, they have proven that for any bounded and continuous functional $F : C([0,T];\Gamma) \to \mathbb{R}$ and $x \in \mathbb{R}^2$

$$\lim_{\epsilon \to 0} \mathbb{E}_x F(\Pi_\epsilon(\cdot)) = \bar{\mathbb{E}}_{\Pi(x)} F(\bar{Y}(\cdot)).$$

The case H has only one critical point

By applying Itô's formula, we have

$$H(X_\epsilon(t)) = H(x) + \frac{1}{2}\int_0^t \Delta H(X_\epsilon(s))\,ds + \int_0^t \nabla H(X_\epsilon(s))dw(s).$$

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Since $X_\epsilon(t)$ rotates many times along the trajectories of H before $H(X_\epsilon(t))$ changes in a sensible way, we expect that for ϵ small

$$\frac{1}{2}\int_0^t \Delta H(X_\epsilon(s))\, ds \sim \int_0^t B(H(X_\epsilon(s)))\, ds,$$

where

$$B(z) = \frac{1}{2T(z)} \oint_{C(z)} \frac{\Delta H(x)}{|\nabla H(x)|}\, dl_z(x).$$

In the same way, for ϵ small

$$\int_0^t |\nabla H(X_\epsilon(s))|^2 \, ds \sim \int_0^t A(H(X_\epsilon(s))) \, ds,$$

where

$$A(z) = \frac{1}{T(z)} \oint_{C(z)} \frac{|\nabla H(x)|^2}{|\nabla H(x)|} \, dl_z = \frac{1}{T(z)} \oint_{C(z)} |\nabla H(x)| \, dl_z.$$

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Therefore, since

$$\int_0^t \nabla H(X_\epsilon(s)) dw(s) = \tilde{w}\left(\int_0^t |\nabla H(X_\epsilon(s))|^2\, ds\right),$$

we can conclude that the slow process $H(X_\epsilon(t))$, for small ϵ approximately is the same as the process governed by the operator

$$\mathcal{L}f(z) = \frac{1}{2}A(z)f''(z) + B(z)f'(z).$$

The general case

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For each $k = 1, \ldots, n$, the differential operator $\bar{\mathcal{L}}_k$, acting on functions f defined on the edge I_k, has the form

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The domain $D(\bar{L})$ is defined as the set of continuous functions on the graph Γ, that are twice continuously differentiable in the interior part of each edge of the graph, and satisfy suitable gluing conditions at the vertices.

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Nevertheless,

it is the generator of a Markov process $\bar{Y}$ on the graph Γ.

In what follows, we shall denote by $\bar{S}(t)$ the semigroup associated with $\bar{Y}$, defined by

$$\bar{S}(t)f(z,k) = \bar{\mathbb{E}}_{(z,k)} f(\bar{Y}(t)),$$

for every bounded Borel function $f : \Gamma \to \mathbb{R}$.

Back to the linear problem

Since the solution of the problem

$$\begin{cases} \dfrac{\partial v_\epsilon}{\partial t}(t,x,y) = \dfrac{1}{2}\,\Delta v_\epsilon(t,x) + \dfrac{1}{\epsilon}\langle \bar{\nabla} H(x), \nabla v_\epsilon(t,x)\rangle, \\ \\ v_\epsilon(0,x) = \varphi(y), \end{cases}$$

is given by

$$v_\epsilon(t,x) = S_\epsilon(t)\varphi(x) = \mathbb{E}_x \varphi(X_\epsilon(t)),$$

in order to study the asymptotics of v_ϵ one would like to use the limit

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But things are more complicated...

Functions defined on Γ and $\mathbb{R}^2$

- For every $u : \mathbb{R}^2 \to \mathbb{R}$ and $(z, k) \in \Gamma$ we have defined

$$u^\wedge(z,k) = \oint_{C_k(z)} u(x)\, d\mu_{z,k}(x),$$

where

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- For every $f : \Gamma \to \mathbb{R}$ and $x \in \mathbb{R}^2$ we have defined

$$f^\vee(x) = f(\Pi(x)) = f(H(x), k(x)).$$

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Notice that

$$u \neq (u^\wedge)^\vee, \qquad f = (f^\vee)^\wedge.$$

The weights in $\mathbb{R}^2$ and Γ

We have assumed that there exists a continuous mapping $\gamma : \Gamma \to (0, +\infty)$ such that

$$\sum_{k=1}^{n} \int_{I_k} \gamma(z,k)\, T_k(z)\, dz < \infty,$$

where, we recall,

$$T_k(z) = \oint_{C_k(z)} \frac{1}{|\nabla H(x)|}\, dl_{z,k}.$$

In particular, this implies that

$$\gamma^\vee \in L^1(\mathbb{R}^2).$$

The weighted L^2 spaces on $\mathbb{R}^2$ and Γ

Once fixed γ, and hence $\gamma^\vee$, we have defined

$$H_\gamma = L^2(\mathbb{R}^2, \gamma^\vee(x)\, dx),$$

and

$$\bar{H}_\gamma = L^2(\Gamma, \nu_\gamma),$$

where the measure ν_γ is defined as

$$\nu_\gamma(A) := \sum_{k=1}^{n} \int_{I_k} \mathbb{I}_A(z,k)\, \gamma(z,k)\; T_k(z)\, dz, \qquad A \subseteq \mathcal{B}(\Gamma).$$

The semigroup $S_\epsilon(t)$ in the weighted space H_γ

We assume that

the semigroup $S_\epsilon(t)$ is well defined on H_γ, for every $\epsilon > 0$.
Moreover, for every fixed $T > 0$, there exists $c_T > 0$ such that

$$\|S_\epsilon(t)\|_{\mathcal{L}(H_\gamma)} \leq c_T, \qquad t \in [0, T], \qquad \epsilon > 0.$$

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$$\|S_\epsilon(t)\|_{\mathcal{L}(H_\gamma)} \leq c_T, \qquad t \in [0, T], \qquad \epsilon > 0.$$

In fact, we have proven that

there exists a strictly positive continuous function
$\gamma : \Gamma \to (0, +\infty)$, that satisfies the condition above and such that

$$\sum_{k=1}^{n} \int_{I_k} \gamma(z, k)\, T_k(z)\, dz < \infty,$$

Convergence of the semigroups

Together with M. Freidlin, I proved that

$$\lim_{\epsilon\to 0}\ \sup_{t\in[\tau,T]}\left|\mathbb{E}_x\, u(X_\epsilon(t)) - \bar{\mathbb{E}}_{\Pi(x)}\, u^\wedge(\bar{Y}(t))\right| = 0, \tag{4}$$

for any $u \in C_b(\mathbb{R}^2)$ and $x \in \mathbb{R}^2$, and for any $0 \leq \tau \leq T$.

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$$\lim_{\epsilon\to 0}\sup_{t\in[\tau,T]}\left|\mathbb{E}_x\, u(X_\epsilon(t))-\bar{\mathbb{E}}_{\Pi(x)}\, u^\wedge(\bar{Y}(t))\right|=0, \tag{4}$$

for any $u\in\ C_b(\mathbb{R}^2)$ and $x\in\ \mathbb{R}^2$, and for any $0\leq\tau\leq\ T$.

This means that

$$\lim_{\epsilon\to 0}\sup_{t\in[\tau,T]}|S_\epsilon(t)u(x)-\bar{S}(t)u^\wedge(\Pi(x))|=0.$$

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In the proof was critical assuming that

$$\frac{dT_k(z)}{dz} \neq 0, \quad (z,k) \in \Gamma.$$

How to prove limit (4)

Limit (4) is not a straightforward consequence of the limit

$$\lim_{\epsilon \to 0} \mathbb{E}_x F(\Pi(X_\epsilon(\cdot))) = \bar{\mathbb{E}}_{\Pi(x)} F(\bar{Y}(\cdot)).$$

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Actually, (4) is a consequence of the following two limits

$$\lim_{\epsilon\to 0} \sup_{t\in[\tau,T]} \left|\mathbb{E}_x\, u(X^\epsilon(t)) - \mathbb{E}_x\, u^\wedge(\Pi(X_\epsilon(t)))\right| = 0, \tag{5}$$

and

$$\lim_{\epsilon\to 0} \sup_{t\in[\tau,T]} \left|\mathbb{E}_x u^\wedge(\Pi(X_\epsilon(t))) - \bar{\mathbb{E}}_{\Pi(x)} u^\wedge(\bar{Y}(t))\right| = 0, \tag{6}$$

that have to be valid for any $0 < \tau < T$ and $x \in \mathbb{R}^2$ and for any $u \in C_b(\mathbb{R}^2)$.

Limit (5)

The limit

$$\lim_{\epsilon \to 0} \sup_{t \in [\tau, T]} \left| \mathbb{E}_x \, u(X^\epsilon(t)) - \mathbb{E}_x \, u^\wedge(\Pi(X_\epsilon(t))) \right| = 0,$$

is a consequence of the following

- an averaging principle in the interior of the edges of the graph Γ;
- precise estimates on the time spent by the process $X_\epsilon(t)$ near the vertices.

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is a consequence of the following

- an averaging principle in the interior of the edges of the graph Γ;
- precise estimates on the time spent by the process $X_\epsilon(t)$ near the vertices.

The proof is delicate and we had to introduce suitable sequences of exit times of the process $X_\epsilon(t)$ from small neighborhoods of the critical points.

Limit (6)

The limit

$$\lim_{\epsilon\to 0} \sup_{t\in[\tau,T]} \left|\mathbb{E}_x u^\wedge(\Pi(X_\epsilon(t))) - \bar{\mathbb{E}}_{\Pi(x)} u^\wedge(\bar{Y}(t))\right| = 0,$$

would be a consequence of

$$\lim_{\epsilon\to 0} \mathbb{E}_x F(\Pi(X_\epsilon(\cdot))) = \bar{\mathbb{E}}_{\Pi(x)} F(\bar{Y}(\cdot)),$$

if for any $u \in C_b(\mathbb{R}^2)$, the function $u^\wedge$ were a continuous function on $\bar{\Gamma}$.

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if for any $u \in C_b(\mathbb{R}^2)$, the function $u^\wedge$ were a continuous function on $\bar{\Gamma}$.

The lack of continuity of $u^\wedge$ at the internal vertices of Γ, requires a more thorough analysis, which also involves estimates of the exit times of $X_\epsilon(t)$ from small neighborhoods of the critical points.

From the SPDE on $\mathbb{R}^2$ to the SPDE on the graph Γ

We are interested in the SPDE

$$\begin{cases} \partial_t u^\epsilon(t,x) = \mathcal{L}_\epsilon u^\epsilon(t,x) + b(u^\epsilon(t,x)) + g(u^\epsilon(t,x))\partial_t \mathcal{W}(t,x), \\ \\ u^\epsilon(0,x) = \varphi(x), \quad x \in \mathbb{R}^2, \end{cases}$$

where

$$\mathcal{L}_\epsilon u(x) = \frac{1}{2}\Delta u(x) + \frac{1}{\epsilon}\langle \bar{\nabla} H(x), \nabla u(x)\rangle, \qquad x \in \mathbb{R}^2.$$

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Our purpose here is to study the

limiting behavior of its unique mild solution u^ϵ in the space $L^q(\boldsymbol{\Omega}; C([0,T]; H_\gamma))$, as $\epsilon \downarrow 0$.

We recall that the noise can be written as

$$\mathcal{W}(t,x) = \sum_{j=1}^{\infty} \widehat{u_j \mu}(x)\, \beta_j(t), \qquad t \geq 0,$$

where μ is the spectral measure of the noise, $\{u_j\}_{j\in\mathbb{N}}$ is an orthonormal basis of $L^2_{(s)}(\mathbb{R}^2, d\mu)$ and $\{\beta_j\}_{j\in\mathbb{N}}$ is a sequence of independent Brownian motions.

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A continuous adapted process $u^\epsilon(t)$, taking values in H_γ is a mild solution to the equation above if

$$u^\epsilon(t) = \quad S_\epsilon(t)\varphi + \int_0^t S_\epsilon(t-s)B(u^\epsilon(s))\,ds$$

$$+ \int_0^t S_\epsilon(t-s)\,G(u^\epsilon(s))\,d\mathcal{W}(s),$$

(see Peszat and Zabczyk, 1997, for the well posedness).

The SPDE on the graph Γ

We introduce now the following SPDE on the graph Γ

$$\begin{cases} \partial_t \bar{u}(t,z,k) = \bar{L}\,\bar{u}(t,z,k) + b(\bar{u}(t,z,k)) + g(\bar{u}(t,z,k))\,\partial_t \bar{\mathcal{W}}(t,z,k), \\ \bar{u}(0,z,k) = \varphi^\wedge(z,k), \quad (z,k) \in \Gamma, \end{cases} \tag{7}$$

where $\bar{L}$ is the generator of the limiting Markov process $\bar{Y}(t)$.

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where $\bar{L}$ is the generator of the limiting Markov process $\bar{Y}(t)$.

The random forcing $\bar{\mathcal{W}}$ is defined by

$$\bar{\mathcal{W}}(t,z,k) = \sum_{j=1}^{\infty} (\widehat{u_j m})^{\wedge}(z,k)\,\beta_j(t), \qquad t \geq 0 \quad (z,k) \in \Gamma.$$

The limit theorem

For any initial condition $\varphi \in H_\gamma$, $q \geq 1$ and $0 < \tau < T$ we have

$$\lim_{\epsilon \to 0} \mathbf{E} \sup_{t \in [\tau, T]} |u_\epsilon(t) - \bar{u}(t)^\vee|^q_{H_\gamma}$$

$$= \lim_{\epsilon \to 0} \mathbf{E} \sup_{t \in [\tau, T]} |u_\epsilon(t)^\wedge - \bar{u}(t)|^q_{H_\gamma} = 0,$$

where u_ϵ and $\bar{u}$ are the unique mild solutions of the SPDE on $\mathbb{R}^2$ and of the SPDE on Γ, respectively.

The case of finite spectral measure

As for $S_\epsilon(t)$, it is possible to show that $\bar{S}(t)$ is well defined in $\bar{H}_\gamma$ and for any $T > 0$

$$\|\bar{S}(t)\|_{\mathcal{L}(\bar{H}_\gamma)} \leq c_T, \qquad t \in [0, T].$$

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Moreover, the noise $\bar{\mathcal{W}}(t)$ takes values in $\bar{H}_\gamma$, so that

the stochastic convolution associated with $\bar{S}(t)$ and $\bar{\mathcal{W}}(t)$ is well defined in $L^2(\mathbf{\Omega}; C([0, T]; \bar{H}_\gamma))$.

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In particular, equation (7) has a unique mild solution $\bar{u}(t)$

$$\bar{u}(t) = \bar{S}(t)\varphi^\wedge + \int_0^t \bar{S}(t-s)B(\bar{u}(s))\,ds + \int_0^t \bar{S}(t-s)\,G(\bar{u}(s))\,d\bar{\mathcal{W}}(s).$$

We have seen that when μ is finite, then

$$\|G(u_1) - G(u_2)\|_{\mathcal{L}_2(RK, H_\gamma)} \leq c\, \|u_1 - u_2\|_{H_\gamma}.$$

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But if we only assume that μ has a density $m \in L^p(\mathbb{R}^2)$, for some $p > 1$, things are more complicated...

The case of infinite spectral measure

Let $G_\epsilon(t,x,y)$ be the kernel corresponding to $S_\epsilon(t)$, i.e.

$$S_\epsilon(t)u(x) = \int_{\mathbb{R}^2} G_\epsilon(t,x,y)u(y)dy, \qquad x \in \mathbb{R}^2.$$

The convergence of $S_\epsilon(t)u(x)$ implies that the kernels $G_\epsilon(t,x,\cdot)$ converge weakly to some $\bar{G}(t,x,\cdot)$, which satisfies

$$\bar{S}(t)^\vee u(x) = \int_{\mathbb{R}^2} \bar{G}(t,x,y)u(y)dy.$$

Together with G. Xi, I have shown that

$$\sup_{\epsilon>0} G_\epsilon(t,x,y) \le \frac{C}{t} \exp\left(-\frac{(\sqrt{H(y)+1}-\sqrt{H(x)+1})^2}{4Ct}\right).$$

In particular, given any compact $K \subset \mathbb{R}^2$, there exist λ_K and R_K d such that for any $t \in (0,\infty)$ and $y \in \mathbb{R}^2$.

$$\sup_{x\in K} G_\epsilon(t,x,y) \le \begin{cases} \dfrac{\lambda_K}{t}, & |y| \le R_K \\ \dfrac{\lambda_K}{t} \exp\left(-\dfrac{|y|^2}{Ct}\right), & |y| > R_K. \end{cases}$$

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Due to the weak convergence of $G_\epsilon(t,x,y)$ to $\bar{G}(t,x,y)$, the same bounds are valid for $\bar{G}(t,x,y)$.

The bounds above allowed us to prove that for each $0 \leq t \leq T$ and $\psi \in H_\gamma$,

$$\sup_{\epsilon>0} \|S_\epsilon(t) M(\psi)\|^2_{\mathcal{L}_2(RK, H_\gamma)} \leq C_T \|m\|_{L^p}\, t^{-(p-1)/p} |\psi|^2_{H_\gamma},$$

where the operator $M(\psi)$ is defined by

$$M(\psi)\xi = \psi\xi.$$

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Moreover,

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for all $0 \leq t \leq T$ and $\psi \in \bar{H}_\gamma$.

In particular, the SPDE on $\mathbb{R}^2$ and the SPDE on Γ are both well-posed.

A more refined limit

A fundamental step was proving that for any $\psi \in H_\gamma$, for any fixed $0 < \tau < T$

$$\lim_{\epsilon \to 0} \sup_{t \in [\tau, T]} \sum_{j=1}^{\infty} \left| \left(S_\epsilon(t) - \bar{S}(t)^\vee \right) (\psi e_j) \right|^2_{H_\gamma} = 0,$$

where $\{e_j\}$ is a complete orthonormal system for the reproducing kernel.

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where $\{e_j\}$ is a complete orthonormal system for the reproducing kernel.

This limit allowed to treat the convergence of stochastic convolutions and conclude that the solutions of the SPDEs on $\mathbb{R}^2$ converge to the solution of the SPDE on Γ.

A weaker type of convergence

One of the key assumptions in order to prove that

$$\lim_{\epsilon\to 0}\sup_{t\in[\tau,T]}\left|\mathbb{E}_x u(X_\epsilon(t)) - \bar{\mathbb{E}}_{\Pi(x)} u^\wedge(\bar{Y}(t))\right| = 0,$$

for any $u \in H_\gamma$ and $0 < \tau < T$, is

$$\frac{dT_k(z)}{dz} \neq 0, \qquad (z,k) \in \Gamma. \tag{8}$$

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Assumption (8) allows to say that if $\alpha \in (4/7, 2/3)$ then for every $u \in C_b^2(\mathbb{R}^2)$ and for every compact set $K \in \mathbb{R}^2$

$$\lim_{\epsilon \to 0} \sup_{x \in K} \left| \mathbb{E}_x u(X_\epsilon(\epsilon^\alpha)) - (u^\wedge)^\vee(x) \right| = 0.$$

What does it happen when (8) is not verified?

We have tried to understand if it is still possible to have some limit in this case, and have proven that for any $0 \leq \tau < T$ and any compact set $K \subset \mathbb{R}^2$,

$$\lim_{\epsilon \to 0} \sup_{x \in K} \left| \int_\tau^T \left[\mathbb{E}_x \varphi(X_\epsilon(t)) - \bar{\mathbb{E}}_{\Pi(x)} \varphi^\wedge(\bar{Y}(t)) \right] \theta(t) dt \right| = 0$$

for any $\varphi \in C_b(\mathbb{R}^2)$ and $\theta \in C_b([\tau, T])$.

The previous limit allowed us to prove that when

$$b = 0, \qquad g = \text{constant}$$

for any fixed $T > 0$, $q \geq 1$ and $\theta \in C([0, T])$

$$\lim_{\epsilon \to 0} \mathbb{E} \left| \int_0^T \left[u_\epsilon(t) - \bar{u}(t)^\vee \right] \theta(t) dt \right|_{H_\gamma}^q$$

$$= \lim_{\epsilon \to 0} \mathbb{E} \left| \int_0^T \left[u_\epsilon(t)^\wedge - \bar{u}(t) \right] \theta(t) dt \right|_{\bar{H}_\gamma}^q = 0.$$

Thank you

Gluing conditions

For any vertex $O_i = (z_i, k_{i_1}) = (z_i, k_{i_2}) = (z_i, k_{i_3})$ there exist finite

$$\lim_{(z,k_{i_j})\to O_i} \bar{L}f(z, k_{i_j}), \qquad j = 1, 2, 3,$$

and the limits do not depend on the edge $I_{k_{i_j}} \sim O_i$. Moreover, for each interior vertex O_i the following gluing condition is satisfied

$$\sum_{j=1}^{3} \pm\alpha_{k_{i_j}}(z_i) d_{k_{i_j}} f(z_i, k_{i_j}) = 0,,$$

where

$$\alpha_k(z) = \oint_{C_k(z)} |\nabla H(x)| \, dl_{z,k}.$$

Here $d_{k_{i_j}}$ is the differentiation along $I_{k_{i_j}}$ and $+$ is taken if the H-coordinate increases along $I_{k_{i_j}}$ and $-$ otherwise.