Liouville theorem for surfaces translating by sub-affine-critical powers of Gauss curvature

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POSTECH

Asia-Pacific Analysis and PDEs Seminar

Liouville theorem

Let $u(x)$ be an entire harmonic ($\Delta u = 0$) function on $\mathbb{R}^n$.

- Bounded (one-side), then $u =$ constant
- If $|u(x)| \leq C(1+|x|)^m$, then $u =$ harmonic polynomials of order $\leq m$

  Examples on $\mathbb{R}^2$,

$$u = 1, \quad r\cos\theta = x, \quad r^2 \cos 2\theta = x^2 - y^2$$

$$r^n \cos(n\theta), \quad r^n \sin(n\theta)$$

Bernstein problem

Let $x_{n+1} = u(x)$ be an entire minimal hypersurface

$$\text{div}\left(\frac{Du}{(1+|Du|^2)^{1/2}}\right) = 0 \text{ on } \mathbb{R}^n.$$

- If $n < 8$, then u must be a linear function
  Bernstein, Fleming, De Giorgi, Almgren, J. Simons,
  Bombieri-De Giorgi-Giusti (counter example when $n = 8$)

Fleming '62 $n = 2$

If $x_{n+1} = u(x)$ is non-flat entire minimal, as $\lambda \to \infty$

$$x_{n+1} = \lambda^{-1} u(\lambda x)$$

converges to non-flat area-minimizing cone $K \subset \mathbb{R}^{n+1}$.

$K^2 \subset \mathbb{R}^3$ is a minimal cone, then K must be flat.

Liouville theorem for Monge-Ampere eq

Consider a convex solution to the equation

$$\det D^2 u = 1 \text{ on } \mathbb{R}^n$$

Example: $u = \frac{1}{2}|x|^2$, or in general

$$u(x) = \frac{1}{2}x^T A x + \mathbf{b} \cdot x + c$$

A: nxn positive matrix with $\det A = 1$, $\mathbf{b}$: vector, c: constant

$$D^2 u = A \quad \Longrightarrow \quad \det D^2 u = 1$$

Celebrated Liouville theorem in MA equation and affine geometry,

Theorem (Jörgens '54, Calabi '58, Pogorelov '72)

There is no other solution except convex quadratic polynomials

In other words, the solutions are unique modulo affine transforms

- Jörgens $n = 2$, Calabi $n \leq 5$, Pogorelov $n \geq 2$
- Cheng-Yau '86 analytic proof based on affine geometry
- Caffarelli-Y.Li '04 and Y.Li-S.Lu '19 perturbation of RHS

$$\det D^2 u = f(x)$$

$f = 1$ outside of compact set, f is $\mathbb{Z}^n$-periodic

Main result

Theorem (C.-Choi-Kim '21)

Classify all possible solutions to

$$\det D^2 u = (1+|Du|^2)^{2-\frac{1}{2\alpha}} \text{ on } \mathbb{R}^2$$

for $0 < \alpha < \frac{1}{4}$.

- $\det D^2 v = (1+|x|^2)^{\frac{1}{2\alpha}-2}$ by Legendre transform

$$v(p) = \sup_x \,[\, p\cdot x - u(x)]$$

- $\alpha = \frac{1}{4}$: affine-critical and $\alpha < \frac{1}{4}$: sub-affine-critical
- RHS and α come from a geometric context

Gauss curvature flow

- Let Σ_t be a 1-parameter family of convex hypersurfaces in $\mathbb{R}^3$.

  Σ_t is α-Gauss Curvature Flow if the surface moves toward inside with the speed K^α. Here

  $$K = \lambda_1 \lambda_2 = \text{ Gaussian curvature at } X \in \Sigma_t.$$

- In other words,

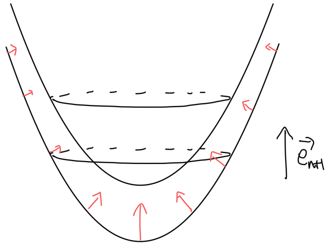
  $$\frac{\partial}{\partial t} X = K^\alpha \mathbf{n}$$

  $\mathbf{n} =$ inward unit normal vector at $X \in \Sigma_t$

$$K \approx \det D^2 u \text{ for graph } x_3 = u(x_1, x_2)$$

A translating soliton is a self-similar solution which moves in constant speed, namely

$$\Sigma_t = \Sigma_0 + t\,\mathbf{e}_3 \quad \text{for } t \in (-\infty, \infty).$$



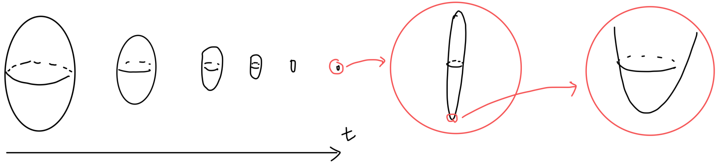
In this caes, Σ_0 solves $K^\alpha = \langle \mathbf{n}, \mathbf{e}_3 \rangle$ and

$$\det D^2 u = (1 + |Du|^2)^{2 - \frac{1}{2\alpha}}$$

Theorem (Andrews '99 - "Fate of Rolling Stones", $n = 2$, $\alpha = 1$)

Every convex closed surface shrinks to a point and becomes round.

- $\alpha = \frac{1}{4}$, the flow converges to an ellipsoid
- Higher dimensions $\alpha > \frac{1}{n+2}$: Guan-Ni, Andrews-Guan-Ni, Brendle-K.Choi-Daskalopoulos
- $\alpha < \frac{1}{4}$, (generic) flow becomes arbitrary elongated and translating solitons are expected to appear



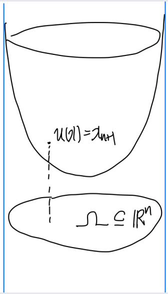
Translators to α-GCF $\alpha > 1/2$

$$\det D^2u = \lambda(1+|Du|^2)^{\frac{n+2}{2}-\frac{1}{2\alpha}} \text{ in } \mathbb{R}^{n+1} \text{ (speed } \lambda > 0\text{).}$$

Theorem (J.Urbas '98 '99)

For $\alpha > 1/2$,

- *Every translator is a graph on some* **convex bounded** *domain $\Omega \subset \mathbb{R}^n$.*
- *Conversely, for given such an $\Omega \subset \mathbb{R}^n$, there is a unique translator which is a complete graph on Ω.*



- Speed λ is given in terms of the area of Ω and α.

Translators $\alpha < 1/2$

- Classification is completely unknown except $\alpha = \frac{1}{n+2}$, affine-critical case
- Rotationally symmetric solution is always entire

Jian-Wang '14 showed

- For $\alpha < \frac{1}{n+1}$, translators are always entire and

$$|x|^{\alpha} \lesssim u(x) \lesssim |x|^{\beta} \text{ for some } 1 < \alpha < \beta$$

- For $\alpha < \frac{1}{2}$, ∞-many non-rotationally symmetric translators exist

Our work shows, for $n = 2$ and $\alpha < \frac{1}{n+2} = \frac{1}{4}$,

$$|x|^{\frac{1}{1-2\alpha}} \lesssim u(x) \lesssim |x|^{\frac{1}{1-2\alpha}}$$

Recall Jörgens' result: upto translations and rotations,

$$u_A(x) = \tfrac{1}{2}(Ax_1^2 + A^{-1}x_2^2) \text{ for some } A > 0$$

solutions are 1-parameter family.

Theorem (CCK, $n = 2$, $\frac{1}{9} \leq \alpha < \frac{1}{4}$)

There is 1-parameter family of translators satisfying

$$u_A(x) = C_\alpha |x|^{\frac{1}{1-2\alpha}} + A|x|^{\gamma_\alpha}\cos(2\theta) + O(|x|^{\gamma_\alpha - \epsilon}), \quad x = (x_1, x_2)$$

and there is no other translator (upto rotations and translations).

- Translators are asymptotically round
- C and γ depend only on α. Moreover, $\gamma \to 2$ as $\alpha \to \frac{1}{4}$

$$r^\gamma \cos 2\theta \to r^2 \cos 2\theta = x_1^2 - x_2^2$$

Why does such a difference appear?

Let $u_0(x)$ be rotationally symmetric translator

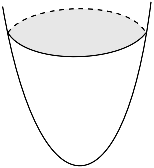
$$\det D^2 u_0 = (1 + |Du_0|^2)^{2-\frac{1}{2\alpha}}.$$

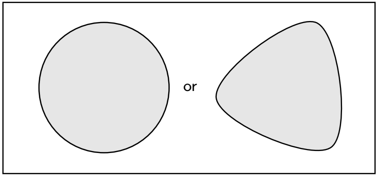
The linearized equation $L_{u_0} w = 0$ has solutions (Jacobi fields) $w = r^\beta \cos 2\theta$ and $r^\beta \sin 2\theta$.

- If $\alpha = \frac{1}{4}$, $\beta = 2$
- If $\alpha < \frac{1}{4}$, $\beta < \frac{1}{1-2\alpha}$

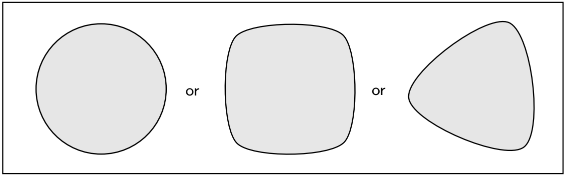
Other Jacobi fields are $w = r^{\frac{2\alpha}{1-2\alpha}} \cos\theta$, $r^{\frac{2\alpha}{1-2\alpha}} \sin\theta$, 1 and so on.

For $\alpha < \frac{1}{9}$, we have more interesting phenomenon





$\frac{1}{16} \leq \alpha < \frac{1}{9}$



$\frac{1}{25} \leq \alpha < \frac{1}{16}$

- When $\alpha = 1/9$, $w = r^{\frac{1}{1-2\alpha}} \cos 3\theta$ is a Jacobi field and asymptotically 3-fold solution bifurcated from there

Theorem (CCK Unique self-similar blow-down)

As $\lambda \to \infty$,

$$u_\lambda(x) := \lambda^{-\frac{1}{1-2\alpha}} u(\lambda x) \to |x|^{\frac{1}{1-2\alpha}} g(\theta)$$

for some $g(\theta)$. The level curve $\{|x|^{\frac{1}{1-2\alpha}} g(\theta) = 1\}$ is a shrinking soliton to $\frac{\alpha}{1-\alpha}$-Gauss curvature flow (of curves) in $\mathbb{R}^2$.

- $u(x)$ is asymptotic to $r^{\frac{1}{1-2\alpha}} g(\theta)$ as $|x| = r \to \infty$
- A closed curve $\Gamma \subset \mathbb{R}^2$ is shrinking soliton for $\frac{\alpha}{1-\alpha}$-curve shortening flow if $K^{\frac{\alpha}{1-\alpha}} = \lambda \langle X, -\mathbf{n} \rangle$
- B. Andrews '03 classified shrinkers as in previous slide

Level curves for $\alpha = \frac{1}{4}$ vs $\alpha < \frac{1}{4}$

- If $\alpha = \frac{1}{4}$, $u(x) = x^T A x = r^2 g(\theta)$ convex paraboloid.

  $\{u(x) = l\}$ are homothetic ellipsoids (=shrinker)

- If $\alpha < \frac{1}{4}$, $u(x) = r^{\frac{1}{1-2\alpha}} g(\theta) + O(r^{\frac{1}{1-2\alpha} - \epsilon})$

  $\{u(x) = l\}$ converges to a shrinker (after rescaling) as $l \to \infty$

Steps of proof

For given translator $u(x)$, show the followings

1. $|x|^{1/1-2\alpha} \lesssim u(x) \lesssim |x|^{1/1-2\alpha}$
2. $u_\lambda(x) := \lambda^{-\frac{1}{1-2\alpha}} u(\lambda x) \to |x|^{\frac{1}{1-2\alpha}} g(\theta)$ along subsequences $\lambda \to \infty$. Here g is unique upto rotations
3. g is actually unique (no rotation) and full convergence holds
4. convergence rate $u(x) = r^{\frac{1}{1-2\alpha}} g(\theta) + O(r^{\frac{1}{1-2\alpha}-\epsilon})$, $\epsilon > 0$
5. $u = u_{\mathbf{y}_0}$ some $\mathbf{y}_0 \in \mathbb{R}^K$. Here, $\{u_{\mathbf{y}}\}_{\mathbf{y}\in\mathbb{R}^K}$ is K-param. family of translators (constructed in the existence part) satisfying the asymptotic condition in previous step

Step 1 and 2 use Daskalopoulos-Savin '08
- showed the homogeneous growth rate $|x|^{\frac{1}{1-2\alpha}}$ (theory of MA eq) and found a sort of monotonicity formula that is crucial in selfsimilar blow-down

Step 3 uses techniques from Allard-Almgren '81
- uniqueness of tangent cones to minimal surface under some integrability assumption

Step 5 uses a nonlinear Gram-Schmidt process is employed to read off correct 'coordinates' $\mathbf{y}_0 \in \mathbb{R}^K$ by use of Merle-Zaag ODE lemma

To sketch step 5, assume the simplest case when u is asymptotically round, i.e.

$$u(x) = c_1|x|^{\frac{1}{1-2\alpha}}(1 + o(|x|^{-\delta})).$$

The relative error $v(x) = \frac{u(x)-c_1|x|^{\frac{1}{1-2\alpha}}}{c_1|x|^{\frac{1}{1-2\alpha}}}$ solves

$$L(v) := r^2 v_{rr} + c_2 r v_r + c_3(v + v_{\theta\theta}) = \mathcal{N}(v)$$

with

$$\mathcal{N}(v) \lesssim (|v| + r|Dv| + r^2|D^2v|)^2.$$

By elliptic regularity,

$$|v| + r|Dv| + r^2|D^2v| = O(r^{-\delta}) \text{ as } r \to \infty$$

making $\mathcal{N}(v)$ negligible compared to Lv.

$Lv = 0$ has the kernel (Jacobi fields)

$$r^{\beta_j^+}\cos(j\theta),\ r^{\beta_j^-}\cos(j\theta),\ r^{\beta_j^+}\sin(j\theta),\ r^{\beta_j^-}\sin(j\theta).$$

Roughly speaking,

$$\{\text{translators}\} \longleftrightarrow \{\text{Jacobi fields with } \beta \in [-\tfrac{1}{1-2\alpha}, 0)\}.$$

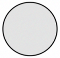
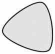
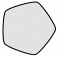
					
$\frac{1}{9} \le \alpha < \frac{1}{4}$	1	X	X	X	$\cdots$
$\frac{1}{16} \le \alpha < \frac{1}{9}$	3	2	X	X	$\cdots$
$\frac{1}{25} \le \alpha < \frac{1}{16}$	5	2	4	X	$\cdots$
$\frac{1}{36} \le \alpha < \frac{1}{25}$	7	2	4	6	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

TABLE 1. the number of parameters modulo rigid motions

Brief idea- nonlinear 1st order ODE system

The eq for v can be rewritten as (with $s = \ln r$)

$$\frac{\partial}{\partial s}\begin{bmatrix} v \\ \partial_s v \end{bmatrix} = \begin{bmatrix} \partial_s v \\ -2\partial_s v - 3v_{\theta\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ \mathcal{N} \end{bmatrix} = \mathbf{L}\begin{bmatrix} v \\ \partial_s v \end{bmatrix} + \begin{bmatrix} 0 \\ \mathcal{N} \end{bmatrix}.$$

Eigenvectors of $\mathbf{L}$ are

$$\mathbf{L}\begin{bmatrix} \sin j\theta \\ \beta_j^{\pm} \sin j\theta \end{bmatrix} = \beta_j^{\pm}\begin{bmatrix} \sin j\theta \\ \beta_j^{\pm} \sin j\theta \end{bmatrix}.$$

Express

$$\begin{bmatrix} v \\ \partial_s v \end{bmatrix}(s) = \sum \varphi_{c,j}^{\pm}(s)\begin{bmatrix} \cos j\theta \\ \beta_j^{\pm} \cos j\theta \end{bmatrix} + \varphi_{s,j}^{\pm}(s)\begin{bmatrix} \sin j\theta \\ \beta_j^{\pm} \sin j\theta \end{bmatrix}$$

and investigate 1st order ODE system of $\{\varphi_{c,j}^{\pm}(s), \varphi_{s,j}^{\pm}(s)\}$.

$\{\varphi^{\pm}_{c,j}(s), \varphi^{\pm}_{s,j}(s)\}$ solves some 'weakly' coupled first order ODE system since $\mathcal{N}$ is much smaller than (v, v_s).

If $\mathcal{N} \equiv 0$, then $\varphi^{\pm}_{c,j}(s) = \varphi^{\pm}_{c,j}(0)e^{\beta^{\pm}_j s}$. So we group eigenspaces into three parts

$\{\beta^{\pm}_j > 0\}$ unstable , $\{\beta^{\pm}_j = 0\}$ neutral , $\{\beta^{\pm}_j < 0\}$ stable

Theorem (Merle and Zaag ODE Lemma '98)

Let $x(t)$, $y(t)$, and $z(t)$ be nonnegative functions such that

$$x + y + z \to 0 \text{ as } t \to \infty$$

and there is $c_0 > 0$ s.t. for all $\epsilon > 0$,

$$x' \geq c_0 x - \epsilon(y + z)$$
$$|y'| \leq \epsilon(x + y + z)$$
$$z' \leq -c_0 z + \epsilon(x + y)$$

for all large time $t > T(\epsilon)$.
Then, as $t \to \infty$, either y is dominant, $x + z = o(y)$, or z is dominant, $x + y = o(z)$.

Liouville results for ancient MCFs and Ricci flows- Angenent, Daskalopoulos, Sesum, K.Choi, Brendle, Haslhofer, Hershkovits, Naff...

Theorem (First leading coefficients)

There are $A_i \in \mathbb{R}$ *such that*

$$v = A_1 e^{\beta_K^+ s} \cos K\theta + A_2 e^{\beta_K^+ s} \sin K\theta + O(e^{(\beta_K^+ - \epsilon)s})$$

Since the eq is nonlinear, the next order asymptotic is not dominated by a Jacobi field. However, the difference of two solutions is dominated by a Jacobi field.

Theorem (Finding next coefficients)

If $v_2 - v_1 = O(e^{\gamma s})$ *with* $\beta_N^+ \leq \gamma < \beta_{N+1}^+ < 0$ *some* N, *then*

$$v_2 - v_1 = A_1 e^{\beta_N^+ s} \cos N\theta + A_2 e^{\beta_N^+ s} \sin N\theta + O(e^{(\beta_N^+ - \epsilon)s})$$

We iterate this and find $v - v_{\mathbf{y}_0} = o(r^{-\frac{1}{1-2\alpha}}) \Rightarrow u \equiv u_{\mathbf{y}_0}$.

Further Problems:
To tackle higher dimension $n = 3$, we need

- the classification of compact shrinking surfaces to sub-affine-critical GCF in $\mathbb{R}^3$
  -Some existence was shown by B. Andrews
- the result of Daskalopoulous-Savin on $\mathbb{R}^3$

When $n = 2$, $\alpha \in (1/4, 1/2)$,

- $\alpha \in (1/4, 1/3) \Rightarrow$ the solutions are entire, but would generically have different growth rates in different axes.
- $\alpha \in (1/3, 1/2) \Rightarrow$ the solutions will not generically be entire