Traveling wave dynamics for a one-dimensional constrained Allen-Cahn equation

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Based on a joint work with

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1. PDEs with non-decreasing constraints

Non-decreasing constraints

In this talk, "non-decreasing constraints on evolution" mean

$$u(x,t) \geq u(x,s) \quad \text{if} \quad t \geq s,$$

or equivalently,

$$\partial_t u(x,t) \geq 0.$$

Background: **Irreversible Phase-field models** (e.g., for brittle fracture)

Let $u = u(x,t)$ denote a displacement field and $z = z(x,t)$ a phase field, which intuitively means

$$z(x,t) = \begin{cases} 1 & \text{if the material is cracked at } (x,t), \\ 0 & \text{if the material is not cracked at } (x,t). \end{cases}$$

Then $t \mapsto z(x,t)$ is supposed to be non-decreasing, unlike $t \mapsto u(x,t)$.

Phase-field model for brittle fracture

The phase-field model reads,

$$
\begin{aligned}
0 &= \operatorname{div}\Big(\alpha_\varepsilon(z)\nabla u\Big) && \text{in } \Omega \times \mathbb{R}_+,\\
\partial_t z &= \Big(\varepsilon \Delta z - \frac{z}{\varepsilon} - \alpha'_\varepsilon(z)|\nabla u|^2\Big)_+ && \text{in } \Omega \times \mathbb{R}_+,
\end{aligned}
$$

which is an irreversible quasi-static evolution, i.e.,

$$
0 = \partial_u \mathcal{F}_\varepsilon(u,z), \quad \partial_t z = \Big(-\partial_z \mathcal{F}_\varepsilon(u,z)\Big)_+,
$$

of the free energy $\mathcal{F}_\varepsilon(u,z)$ (= Ambrosio-Tortorelli regularization of the Francfort-Marigo energy) given by

$$
\mathcal{F}_\varepsilon(u,z) = \frac{1}{2}\int_\Omega \alpha_\varepsilon(z)|\nabla u|^2 \,\mathrm{d}x + \int_\Omega \Big(\frac{\varepsilon}{2}|\nabla z|^2 + \frac{1}{2\varepsilon} z^2\Big)\,\mathrm{d}x.
$$

[Frémond-Nedjar ’96], [Bonetti-Schimperna ’04], [Mielke-Roubíček ’08], [Knees-Rossi-Zanini ’13-], [Takaishi-Kimura ’09,’11],...,[A-Schimperna ’21]

Gradient flows with non-decreasing constraints

Let us focus on the gradient flow structure with non-decreasing constraints, roughly speaking,

$$\partial_t u(t) = \Big(-\partial J(u(t)) \Big)_+ \geq 0$$

for some (possibly non-convex) functional $J : X \to \mathbb{R}$, say $X = L^2(\Omega)$.
Equivalently, it can be rewritten as

$$\partial_t u(t) = -\partial J(u(t)) - \mu,$$

where μ can be characterized as

$$\mu \in \partial I_{[0,+\infty)}(\partial_t u) \quad \text{and} \quad \mu = -\Big(-\partial J(u(t)) \Big)_- .$$

Stabilization $\leftrightarrow$ Constraint

Aim of this talk

Address ourselves onto simpler PDEs with non-decreasing constraints and discuss asymptotic behavior of solutions.

In particular, we shall discuss traveling wave dynamics for the 1D Allen-Cahn equation with non-decreasing constraints,

$$(*) \qquad u_t = \Big(u_{xx} - W'(u)\Big)_+ \quad \text{in } \mathbb{R} \times \mathbb{R}_+.$$

cf.) [A-Efendiev '19] Cauchy-Dirichlet problem in bounded domains of $\mathbb{R}^N$

- well-posedness, (partial) smoothing effect
- energy-dissipation estimates, absorbing set, global attractor
- reformulation of $(*)$ as an obstacle problem
- convergence to equilibria as $t \to +\infty$ and steady-state equation

2. Traveling wave dynamics

Allen-Cahn equation in $\mathbb{R}$

Let us recall the Cauchy problem for the classical Allen-Cahn equation,

$$\text{(AC)} \begin{cases} u_t = u_{xx} - f(u) & \text{in } \mathbb{R} \times \mathbb{R}_+, \\ u|_{t=0} = u_0 & \text{in } \mathbb{R}, \end{cases}$$

where f satisfies

$$\begin{cases} f(a_\pm) = f(a_0) = 0, \quad f'(a_\pm) > 0, \\ f > 0 \ \text{in } (a_-, a_0), \quad f < 0 \ \text{in } (a_0, a_+), \end{cases}$$

for some $a_- < a_0 < a_+$, i.e., $f = W'$ with a double-well potential W.

Allen-Cahn equation in $\mathbb{R}$

Let us recall the Cauchy problem for the classical Allen-Cahn equation,

$$\text{(AC)} \begin{cases} u_t = u_{xx} - f(u) & \text{in } \mathbb{R} \times \mathbb{R}_+, \\ u|_{t=0} = u_0 & \text{in } \mathbb{R}. \end{cases}$$

♠ phase separation model (e.g., binary alloy),

♠ L^2 gradient flow of the free energy

$$J(u) := \frac{1}{2} \int_{\mathbb{R}} |\partial_x u(x)|^2 \, \mathrm{d}x + \int_{\mathbb{R}} W(u(x)) \, \mathrm{d}x$$

where $W' = f$. Namely,

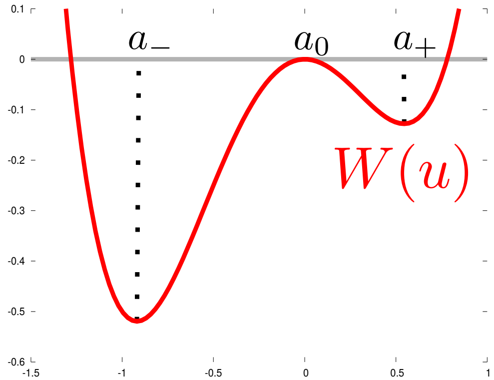
$$\text{(AC)} \;\Leftrightarrow\; u_t = -\partial J(u) \;\text{ in } L^2(\mathbb{R}), \quad t > 0.$$

Hence $t \mapsto J(u(t))$ is non-increasing.

Traveling wave solutions for (AC)

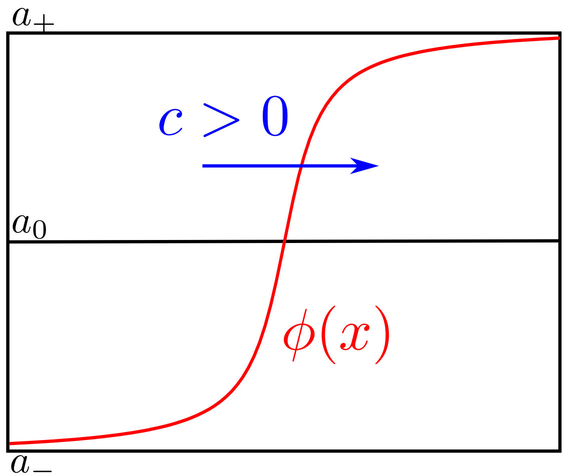
The traveling wave solution $u(x,t) = \phi(x - ct)$ is characterized by a profile function ϕ and a velocity constant c satisfying

$$\begin{cases} -c\phi' = \phi'' - f(\phi) & \text{in } \mathbb{R}, \\ \phi(\xi) \to a_\pm & \text{as } x \to \pm\infty \end{cases}$$



and

$$c \begin{cases} < 0 & \text{if } W(a_+) < W(a_-), \\ = 0 & \text{if } W(a_-) = W(a_+), \\ > 0 & \text{if } W(a_-) < W(a_+). \end{cases}$$



Traveling wave solutions for (AC)

[Fife-McLeod '72]

- Existence and "uniqueness" of TWs

  Phase-plain analysis

- Exponential stability of TWs: if

$$u_0(x) \in [a_-, a_+], \quad \limsup_{x \to -\infty} u_0(x) < a_0, \quad \liminf_{x \to +\infty} u_0(x) > a_0,$$

then there exist constants $x_0 \in \mathbb{R}$, $K, \kappa > 0$ such that

$$\|u(\cdot, t) - \phi(\cdot - ct - x_0)\|_{L^\infty(\mathbb{R})} \le K \mathrm{e}^{-\kappa t} \text{ for all } t \ge 0.$$

Schauder estimate, precompactness of $\{u(\cdot + ct, t) \colon t \ge 0\}$,
sub / supersolution method

[X. Chen '92] Non-local reaction, Sub / supersolution method <u>only</u>

Constrained Allen-Cahn equation

Now, we shall consider

$$(\mathrm{AC})_+ \begin{cases} u_t = \Big(u_{xx} - f(u)\Big)_+ & \text{in } \mathbb{R} \times \mathbb{R}_+, \\ u\big|_{t=0} = u_0 & \text{in } \mathbb{R}, \end{cases}$$

where f satisfies

$$\begin{cases} f(a_\pm) = f(a_0) = 0, \quad f'(a_\pm) > 0, \\ f > 0 \ \text{ in } (a_-, a_0), \quad f < 0 \ \text{ in } (a_0, a_+) \end{cases}$$

for some $a_- < a_0 < a_+$ and

$$(s)_+ := \max\{s, 0\} \geq 0.$$

Constrained Allen-Cahn equation

Now, we shall consider

$$(\mathrm{AC})_+ \begin{cases} u_t = \Big(\underbrace{u_{xx} - f(u)}_{= -\partial J(u)}\Big)_+ & \text{in } \mathbb{R} \times \mathbb{R}_+, \\ u|_{t=0} = u_0 & \text{in } \mathbb{R}, \end{cases}$$

Then $t \mapsto J(u(t))$ is still non-increasing.

Moreover, $(\mathrm{AC})_+$ is equivalent to

- $u_t = -\partial J(u) - \mu, \quad \mu \in \partial I_{[0,+\infty)}(u_t), \quad u|_{t=0} = u_0,$
- $\min\{u - u_0\,,\, u_t - u_{xx} + f(u)\} = 0, \quad u|_{t=0} = u_0$

  $\Leftrightarrow \quad u_t = -\partial J(u) - \mu, \quad \mu \in \partial I_{[u_0(x),\infty)}(u), \quad u|_{t=0} = u_0.$

Traveling wave dynamics for $(\mathrm{AC})_+$

We shall discuss

- existence and uniqueness of traveling wave solutions,
- convergence to a traveling wave solution.

In what follows, we may restrict ourselves to a balanced potential:

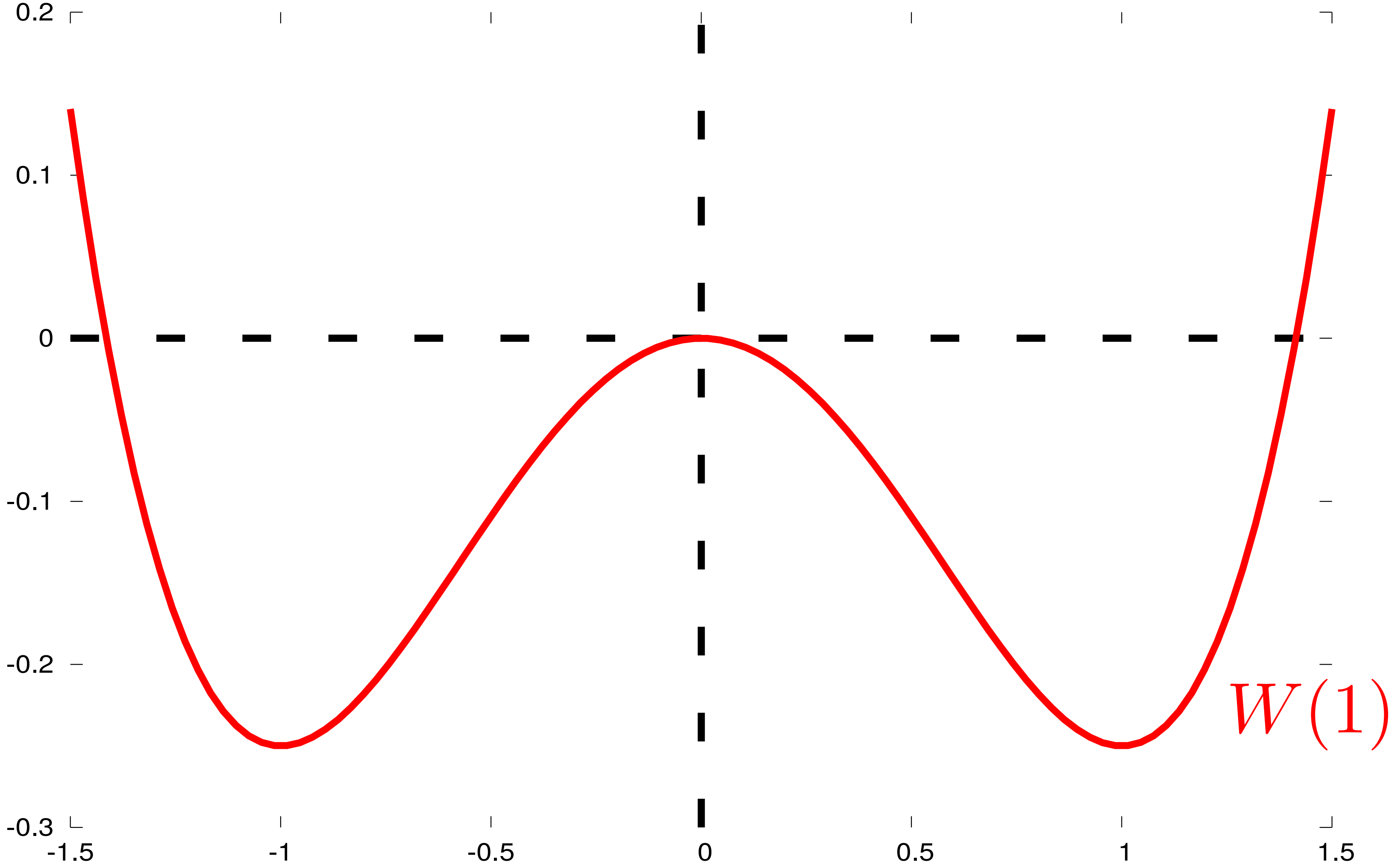
$$a_\pm = \pm 1, \quad a_0 = 0, \quad W(u) = \frac{1}{4}(u^2 - 1)^2,$$

for which the TW of (AC) fulfills

$$c = 0 \quad \text{and} \quad \phi(\pm\infty) = \pm 1.$$

3. Existence and uniqueness of traveling waves

Balanced double-well potential



Balanced double-well potential $W(u)$

Heuristic construction

Substitute $u(x,t) = \phi(x - ct)$ to $(\mathbf{AC})_+$. Then a profile equation reads,

$$-c\phi' = \Big(\phi'' - f(\phi)\Big)_+.$$

How can we solve it ?

Instead, we shall derive an alternative profile equations...

Heuristic construction

Suppose that $u(-\infty) \equiv \alpha \in (-1, 0)$, which is a steady-state for $(\text{AC})_+$.

(i) Due to the non-decreasing constraint $u_t \geq 0$,

$$u(x,t) \geq \alpha.$$

Then the region $[u < \alpha]$ on the graph of $W(u)$ is prohibited.

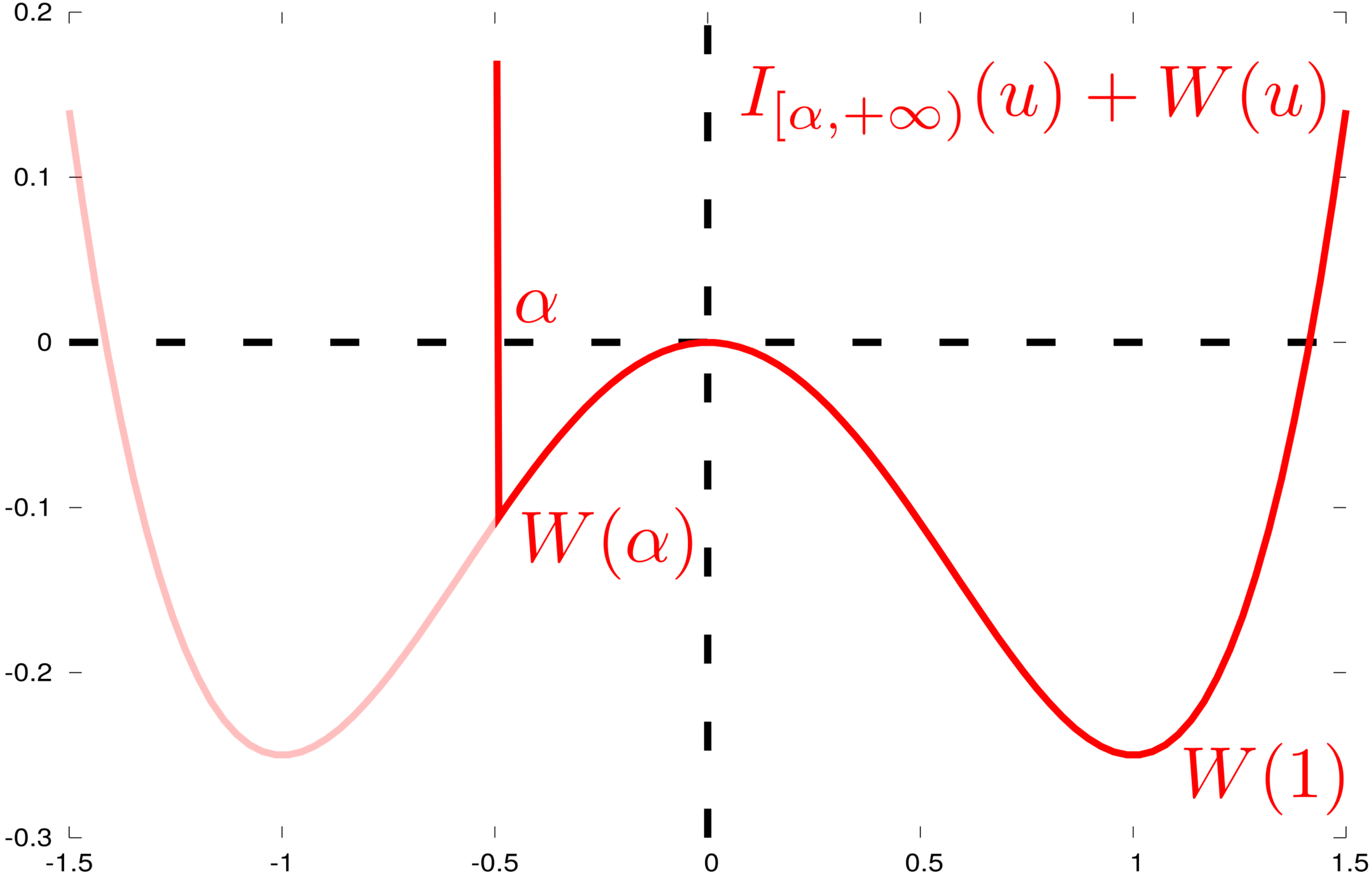
(ii) Analogously to the reformulation, u may solve the obstacle problem,

$$u_t - u_{xx} + \underbrace{f(u)}_{= W'(u)} + \partial I_{[\alpha,+\infty)}(u) \ni 0 \quad \text{in } \mathbb{R} \times \mathbb{R}.$$

Roughly speaking,

$$W'(u) + \partial I_{[\alpha,+\infty)}(u) = \partial\big(W + \underbrace{I_{[\alpha,+\infty)}}_{\text{truncation !}}\big)(u).$$

Heuristic construction



Truncated balanced double-well potential $W(u)$

Profile equation for traveling waves

Let $\alpha \in (-1, 0)$ and substitute $u = \phi(x - ct)$. Then

$$(*) \qquad -c\phi' - \phi'' + \underbrace{W'(\phi) + \partial I_{[\alpha,\infty)}(\phi)}_{=\partial(W+I_{[\alpha,\infty)})(\phi)} \ni 0 \quad \text{in } \mathbb{R}$$

is derived along with

$$\phi(-\infty) = \alpha \quad \text{and} \quad \phi(+\infty) = 1 \quad \text{and} \quad \phi' \geq 0$$

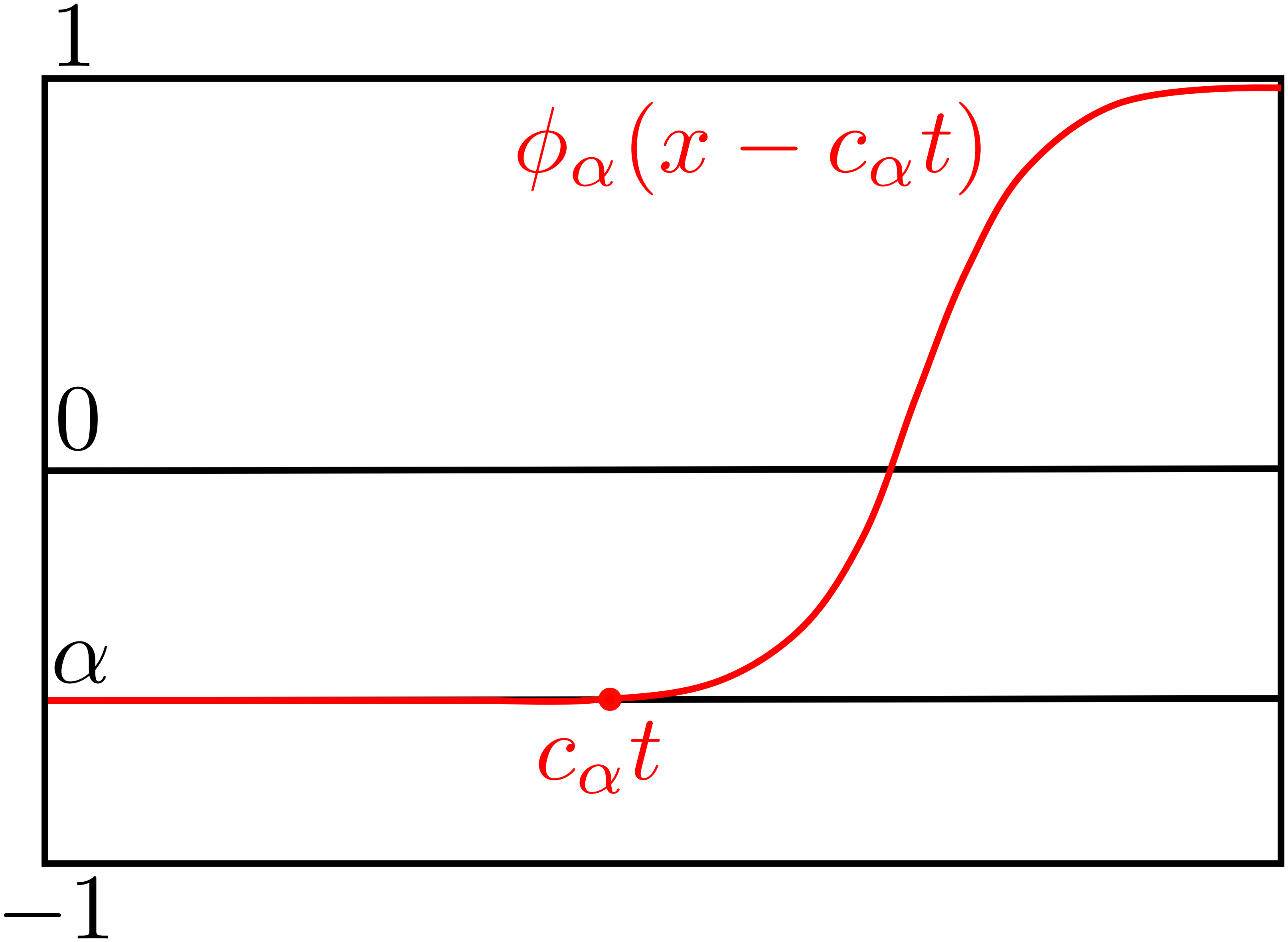
as a profile equation.

Indeed, let ϕ_α be a solution of $(*)$ for some $c = c_\alpha$.

Then $u(x, t) = \phi_\alpha(x - c_\alpha t)$ solves $(\mathbf{AC})_+$

and $c_\alpha < 0$ if $\alpha \neq -1$.

Existence of traveling wave solutions



Existence of traveling wave solutions

Theorem 1 (Traveling wave solutions [A-K-N])

For each $\alpha \in (-1, 0)$, $(\mathbf{AC})_+$ has a solution $u(x,t) = \phi_\alpha(x - c_\alpha t)$ for some profile function $\phi_\alpha(\xi)$ and a velocity constant c_α satisfying

$$\lim_{\xi \to +\infty} \phi_\alpha(\xi) = 1, \quad \lim_{\xi \to -\infty} \phi_\alpha(\xi) = \alpha.$$

Moreover, it holds that

(i) $-c\phi_\alpha' - \phi_\alpha'' + f(\phi_\alpha) + \partial I_{[\alpha,\infty)}(\phi_\alpha) \ni 0$ in $\mathbb{R}$,

$\Leftrightarrow \quad \min\left\{\phi_\alpha - \alpha \,,\, -c_\alpha\phi_\alpha' - \phi_\alpha'' + f(\phi_\alpha)\right\} = 0$ in $\mathbb{R}$,

(ii) $\phi_\alpha \in W^{2,\infty}(\mathbb{R})$, $\phi_\alpha' \in L^2(\mathbb{R})$,

(iii) $\alpha \leq \phi_\alpha < 1$, $0 \leq \phi_\alpha' < +\infty$ in $\mathbb{R}$, $-\infty < c_\alpha < 0$,

Existence of traveling wave solutions

Theorem 1 (Traveling wave solutions [A-K-N] (contd.))

There exists $s_0 \in \mathbb{R}$ such that

$$\phi_\alpha(s) = \alpha \text{ on } (-\infty, s_0], \quad \phi_\alpha(s) > \alpha \text{ on } (s_0, \infty).$$

Furthermore, $-c_\alpha \phi_\alpha' - \phi_\alpha'' + f(\phi_\alpha) = 0$ in $(s_0, +\infty)$, and hence,

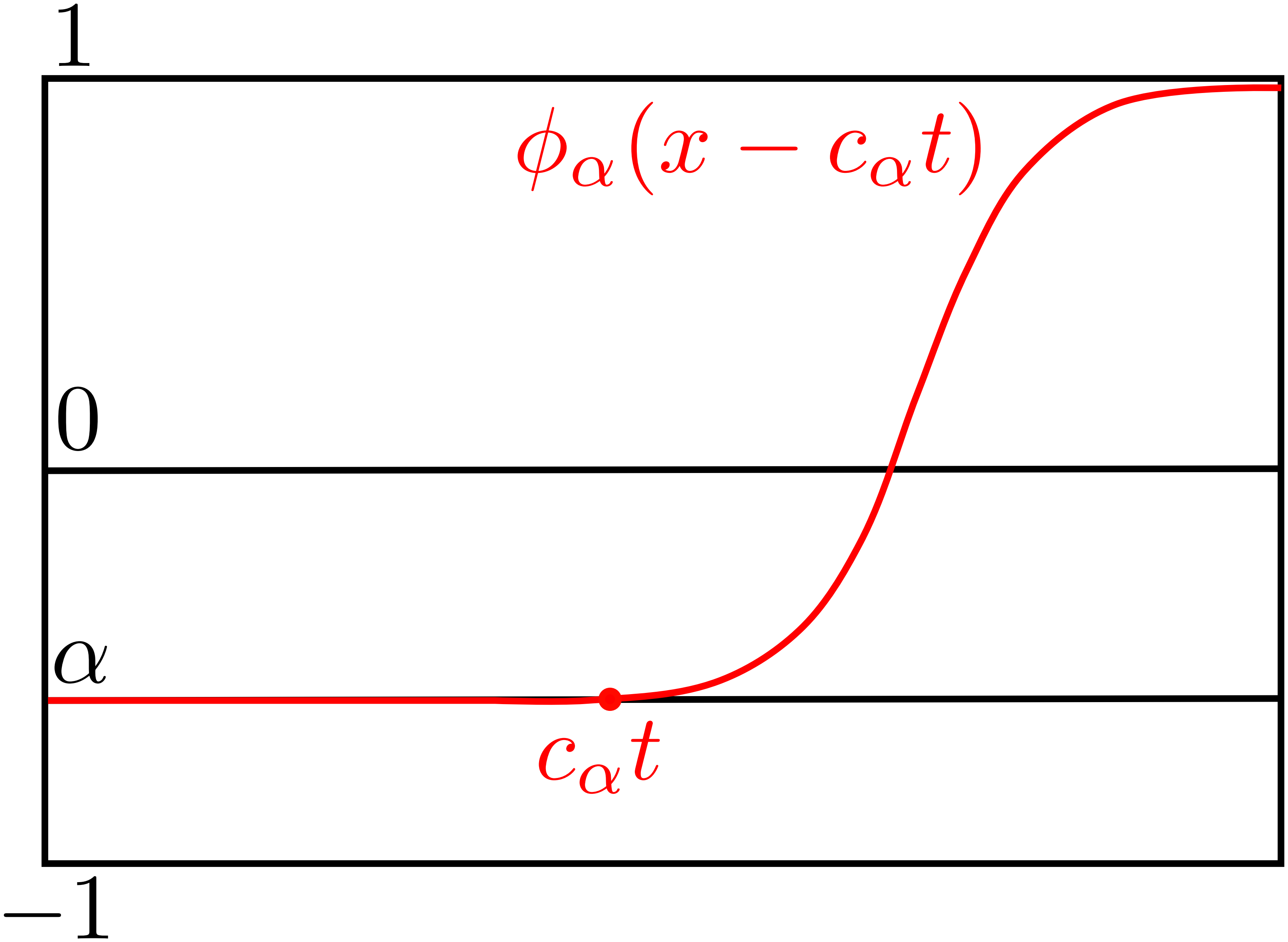
$$\phi_\alpha''(s_0 - 0) = 0 \text{ and } \phi_\alpha''(s_0 + 0) = f(\alpha) > 0,$$

which implies $\phi \notin C^2(\mathbb{R})$.

In what follows, we set $s_0 = 0$ by translation. Hence

$$\phi_\alpha = \alpha \text{ on } (-\infty, 0], \quad \phi_\alpha > \alpha \text{ on } (0, \infty).$$

Existence of traveling wave solutions



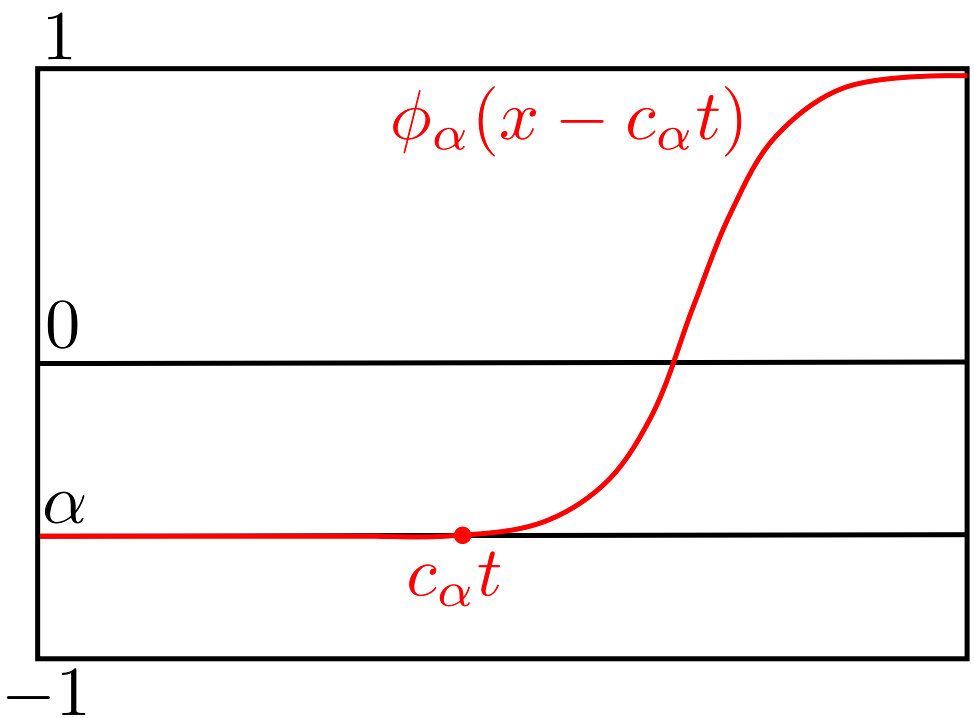
Uniqueness of profile and velocity

Theorem 2 (Uniqueness of traveling waves [A-K-N])

Concerning traveling wave solutions discussed in Theorem 1, it holds that

(i) the velocity c_α is unique for each α,

(ii) the profile function ϕ_α is unique (up to translation) for each α.



4. Convergence to traveling waves

Question

Question

Can we also prove the (exponential) convergence of solutions for $(\mathbf{AC})_+$ to traveling waves as in the classical Allen-Cahn equation ?

We cannot expect stability of traveling wave solutions.

Proposition 3 (Instability of traveling waves [A-K-N])

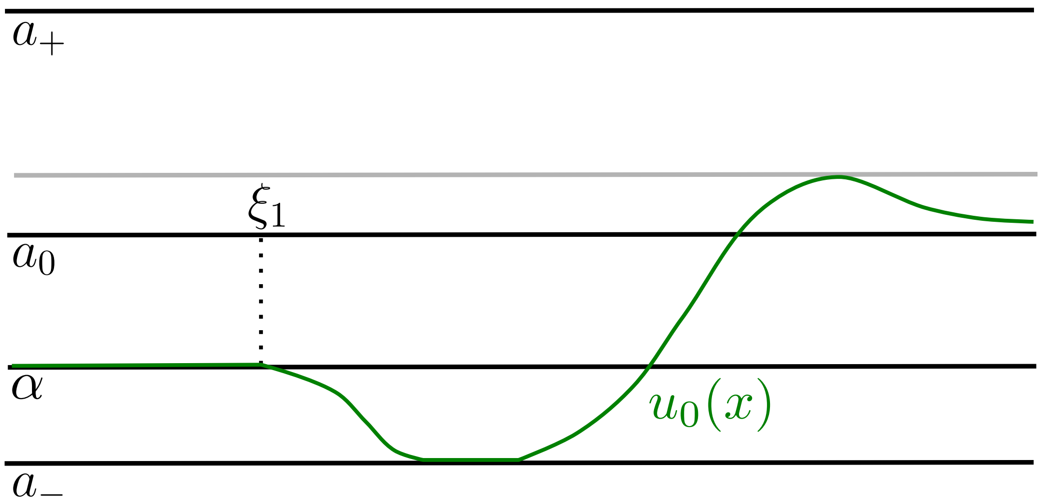
For each $\alpha \in (a_-, a_0)$, the traveling wave solution $\phi_\alpha(x - c_\alpha t)$ of $(\mathrm{AC})_+$ is unstable in $L^\infty(\mathbb{R})$.

Hence the basin of attraction of the traveling wave solution for $(\mathrm{AC})_+$ is smaller than those for (AC).

Hypotheses for initial data

For $\alpha \in (a_-, a_0)$, we assume (with $a_\pm = \pm 1$ and $a_0 = 0$):

$$(\mathbf{H})_\alpha \begin{cases} u_0 \in H^2_{\mathrm{loc}}(\mathbb{R}), \quad \liminf_{x \to +\infty} u_0(x) > a_0, \\ a_- \le \inf_{x \in \mathbb{R}} u_0(x) \le \sup_{x \in \mathbb{R}} u_0(x) < a_+, \\ u_0 \equiv \alpha \ \text{ on } (-\infty, \xi_1] \ \text{ for some } \xi_1 \in \mathbb{R}. \end{cases}$$



Then one can define

$$r(t) := \sup\{r \in \mathbb{R} \colon u(x,t) = \alpha \ \text{ for all } \ x \le r\} \in \mathbb{R} \ \text{ for } \ t \ge 0.$$

Main result

Theorem 4 (Exponential convergence to a TW [A-K-N])

Let $\alpha \in (a_-, a_0)$ be such that $f'(\alpha) > 0$ and assume that u_0 satisfies $(\mathbf{H})_\alpha$. Let $u = u(x,t)$ be the L^2_{loc} solution to $(\mathbf{AC})_+$ for the initial datum u_0. Then there exist $x_0 \in \mathbb{R}$ and $K, \kappa > 0$ such that

$$\|u(\cdot,t) - \phi_\alpha(\cdot - c_\alpha t - x_0)\|_{L^\infty(\mathbb{R})} \leq K\mathrm{e}^{-\kappa t} \quad \text{for all } t \geq 0.$$

Set $r(t) := \sup\{r > 0 \colon u(\cdot,t) \equiv \alpha \text{ on } (-\infty, r]\}$. Then

$$|r(t) - c_\alpha t - x_0| \lesssim \mathrm{e}^{-\frac{\kappa}{2}t} \quad \text{as} \quad t \to +\infty.$$

Supersolutions of $(\mathbf{AC})_+$ cannot decrease !

We note that

$$U_t \geq \Big(U_{xx} - f(U)\Big)_+ \geq 0$$

implies

$U(x,t)$ cannot decrease in time !

Hence the sub- and supersolution method does not work well for $(\mathbf{AC})_+$ (cf. Fife-McLeod & Chen).

$\Rightarrow$ Our proof consists of “4 phases”.

Rough outline of proof

Phases 1 & 2 Reduction to a simplified system

Reduction to a constant obstacle problem

There exists $t_1 > 0$ such that, for all $t \geq t_1$,

$$u(x,t) = u_0(x) \quad \text{if and only if} \quad u(x,t) = \alpha.$$

Hence $(\mathbf{AC})_+$ is reduced to

$$(\mathbf{AC})_\alpha \quad \min\{u - \alpha\,,\; u_t - u_{xx} + f(u)\} = 0 \;\text{ in } \mathbb{R}.$$

Phase 3 Quasi-convergence of the orbit $\mathcal{O} = \{u(\,\cdot\, + c_\alpha t, t) \colon t \geq t_1\}$ to the limit $\phi_\alpha(\,\cdot\, - x_0)$ unif. in $\mathbb{R}$ for some $x_0 \in \mathbb{R}$ **Energy method**

Phase 4 Exponential convergence of $u(\cdot, t)$ to $\phi_\alpha(\cdot - c_\alpha t - x_0)$ uniformly in $\mathbb{R}$ as $t \to +\infty$ **Sub- and supersolution method**

Initial Phase

Claim

$$\exists t_1 > 0 \; ; \; \inf_{x \in \mathbb{R}} u(x, t_1) > a_-.$$

Let u_{ac} be the unique solution to (AC) with the same datum u_0. Moreover,

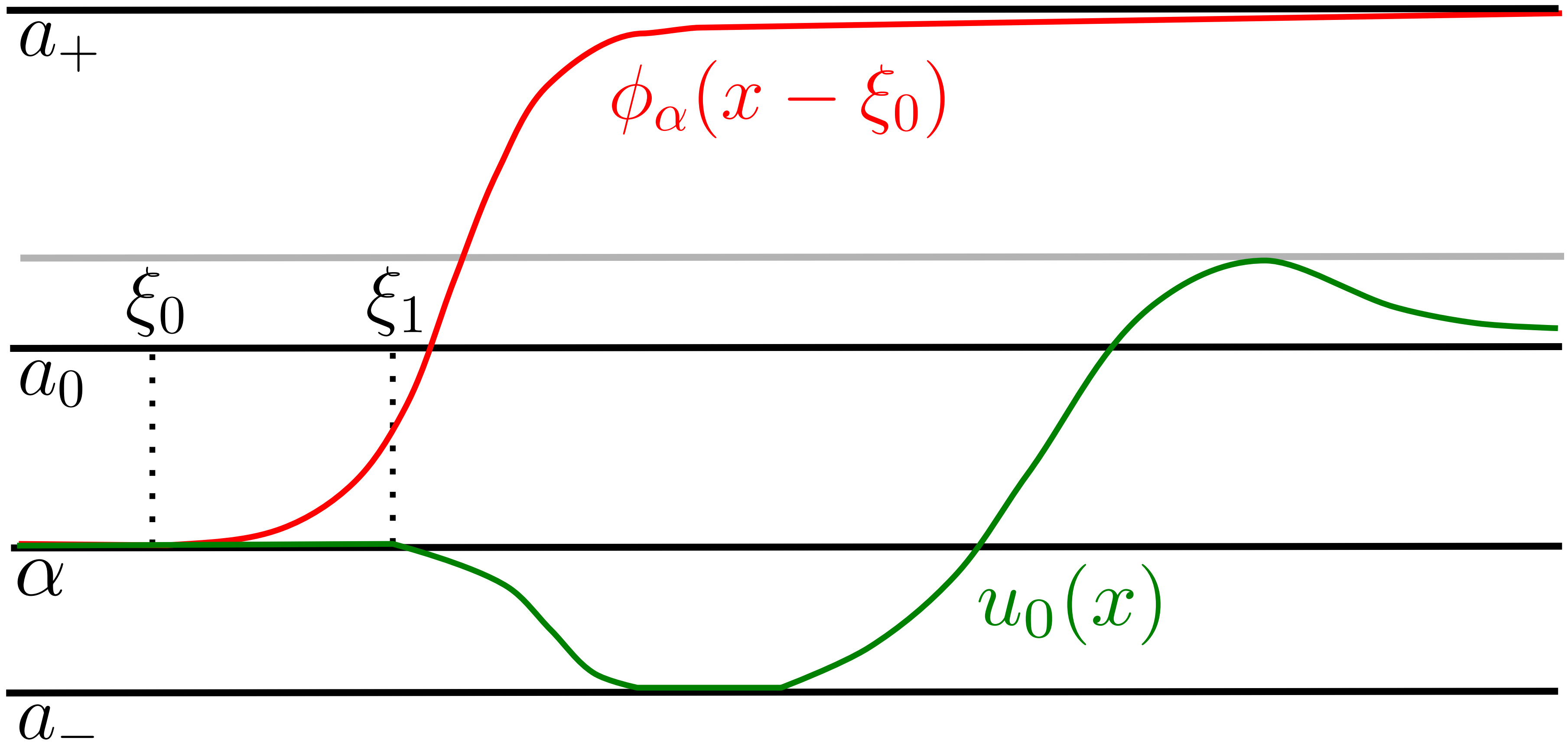
$$u_t = \Big(u_{xx} - f(u)\Big)_+ \geq u_{xx} - f(u) \text{ in } \mathbb{R} \times \mathbb{R}_+.$$

By comparison principle,

$$u(x,t) \geq u_{ac}(x,t) \to \phi_{ac}(x) \text{ as } t \to +\infty,$$

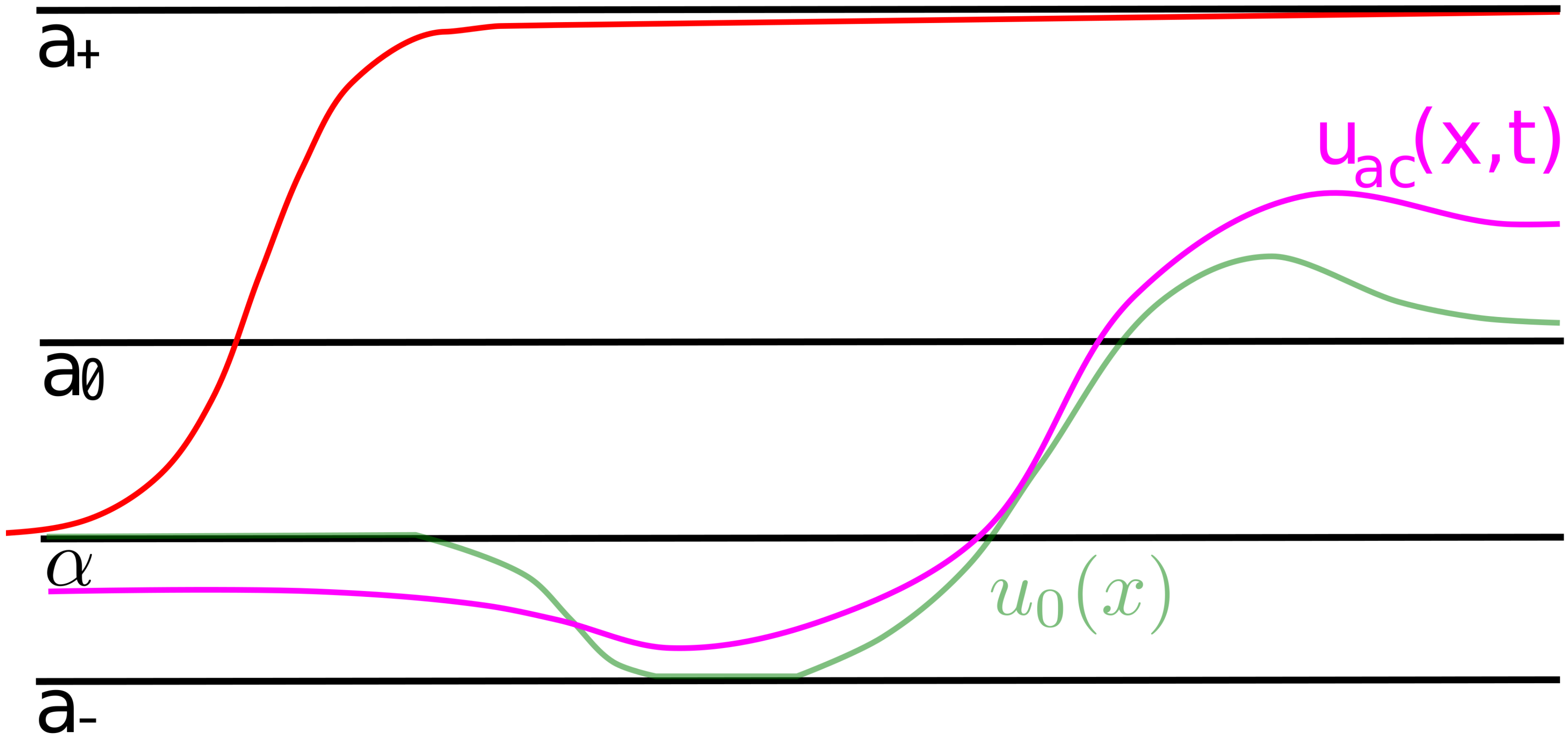
where ϕ_{ac} is a TW with $c = 0$ connecting $a_\pm$ at $x = \pm\infty$.

Initial Phase



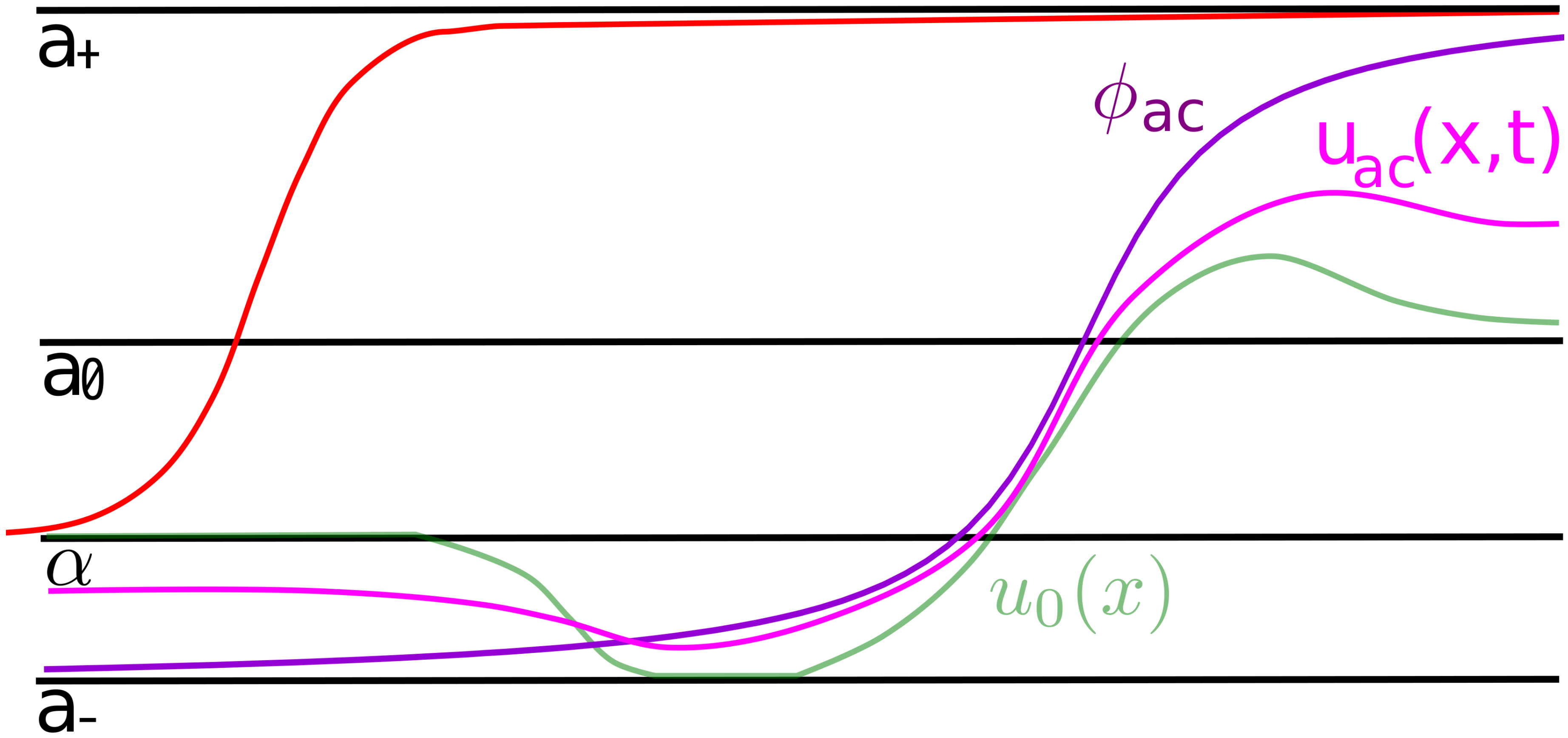
Let $u_{ac}(x, t)$ be the solution to (AC) with $u_{ac}(x, 0) = u_0(x)$.

Initial Phase



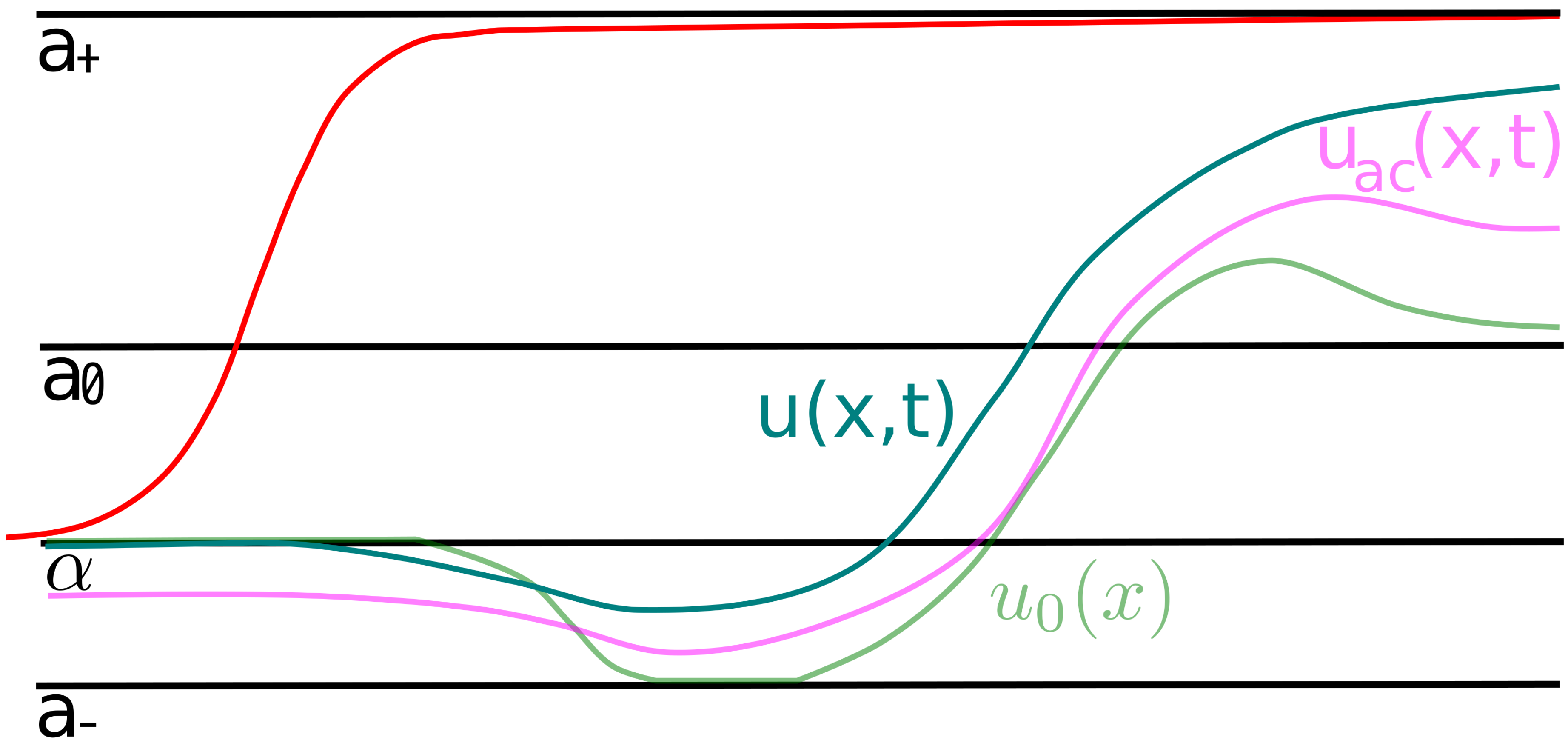
Let $u_{ac}(x,t)$ be the solution to (AC) with $u_{ac}(x,0) = u_0(x)$.

Initial Phase



Then $u_{ac}(x,t)$ converges to a layer solution $\phi_{ac}(x)$ (with $c = 0$).

Initial Phase



The solution $u(x,t)$ of $(\textbf{AC})_+$ is a supersolution to (**AC**).
Hence $u(x,t) \geq u_{ac}(x,t)$.

Second Phase

Lemma 5 (Reduction to a constant obstacle problem)

There exists $t_2 > t_1$ such that, for all $t > t_2$,

$$u(x,t) = u_0(x) \quad \Leftrightarrow \quad u(x,t) = \alpha.$$

We employ a subsolution to $(\mathbf{AC})_+$ given by

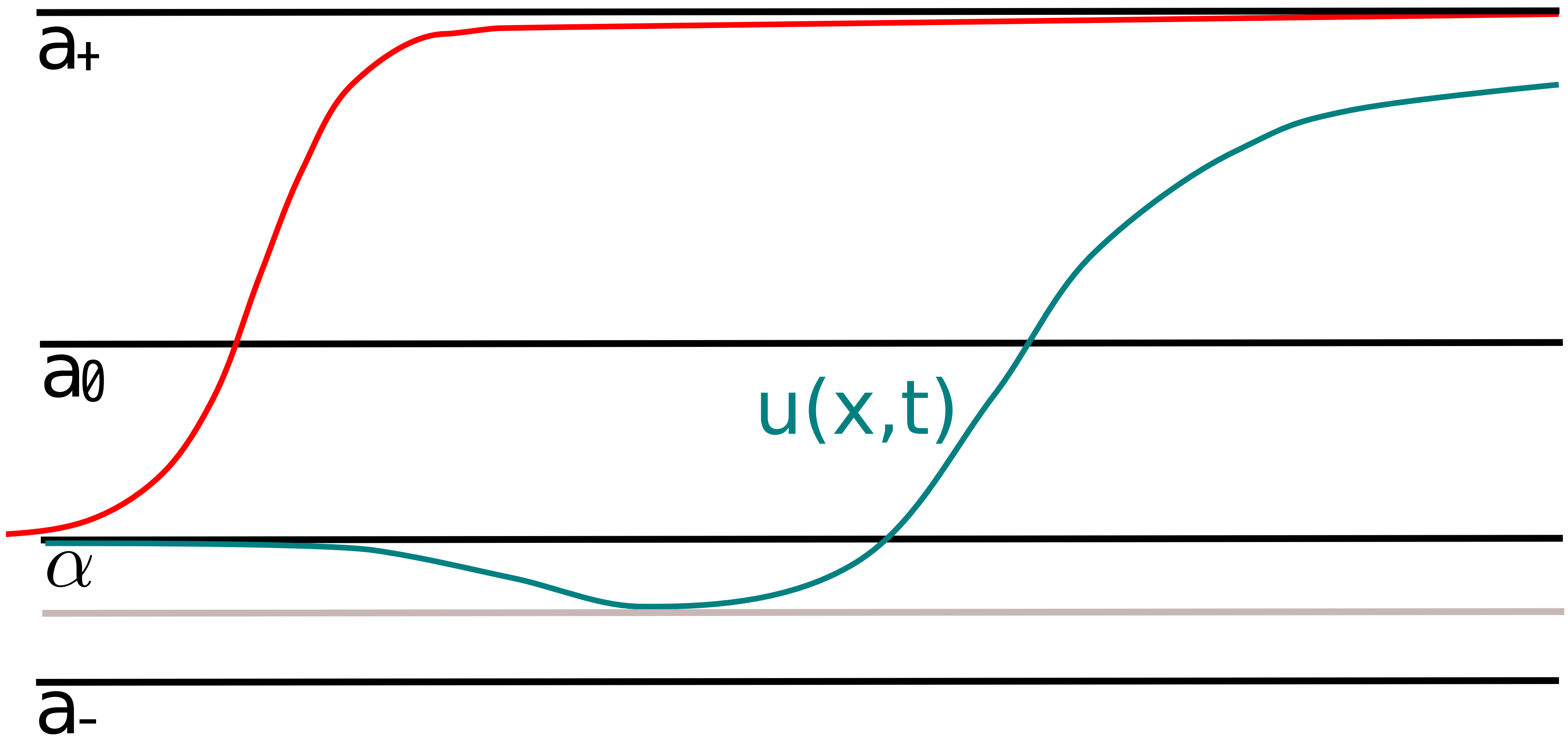
$$U_\gamma(x,t) := \phi_\gamma(x - c_\gamma t - \sigma\delta(1 - \mathrm{e}^{-\beta t}) - h^-) - \delta \mathrm{e}^{-\beta t}$$

for some $h^- \in \mathbb{R}$, $\beta, \delta, \sigma > 0$ and $\gamma \in (a_-, a_0)$ satisfying

$$a_- < \gamma - \delta < \inf_{x \in \mathbb{R}} u(x, t_1).$$

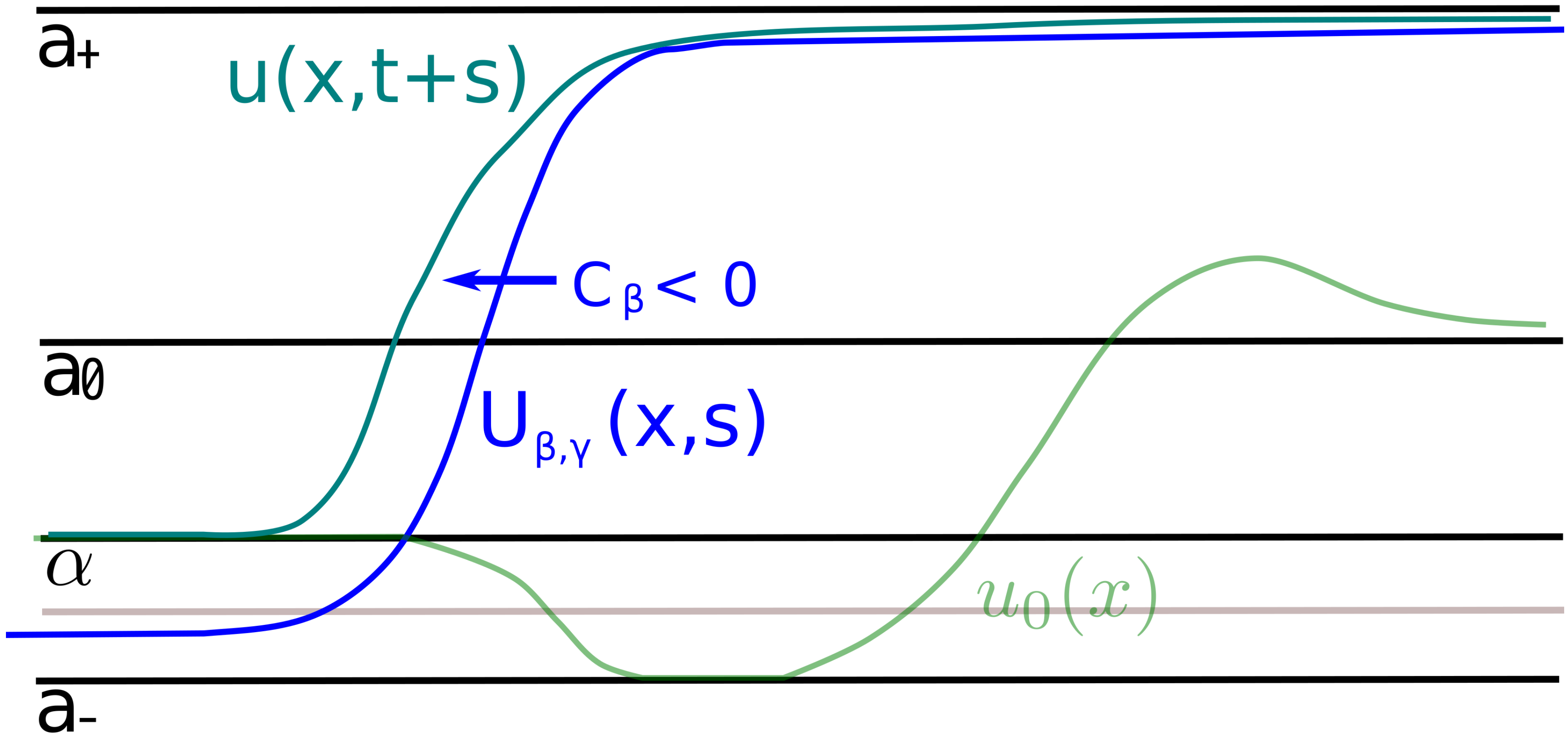
Then we assure that $c_\gamma < 0$.

Second Phase



$$\inf_{x\in\mathbb{R}} u(x,t_1) > a_-$$

Second Phase



Eventually, $u(x, t+s) \geq U_{\beta,\gamma}(x, s) > \alpha = u_0(x)$ for all $x \geq \xi_1$.

Therefore $u(x, t) = u_0(x) \iff u(x, t) = \alpha$.

Second Phase

Thanks to the reformulation, we assure that

$$(\textbf{AC})_+ \;\Leftrightarrow\; (\textbf{AC})_{\alpha} \;\; \min\{u - \alpha, u_t - u_{xx} + f(u)\} = 0 \;\; \text{in } \mathbb{R}$$

for all $t > t_2$.

♠ Third Phase

Lemma 6 (Quasi-convergence of $u(\cdot - c_\alpha t, t)$)

There exist a sequence $t_n \to +\infty$ and $\xi \in \mathbb{R}$ such that

$$\|u(\cdot, t_n) - \phi_\alpha(\cdot - c_\alpha t_n + \xi)\|_{L^\infty(\mathbb{R})} \to 0.$$

We prove the precompactness of $\{u(\cdot - c_\alpha t, t) \colon t > t_2\}$ in $H^1_{\mathrm{loc}}(\mathbb{R})$ by developing local energy estimates for $(\mathbf{AC})_\alpha$ and identify the limit of the orbit.

Thereby, for any $\delta > 0$ (small), one can take $n_\delta \in \mathbb{N}$ such that

$$\phi_\alpha(x - c_\alpha t_{n_\delta}) - \delta \leq u(x, t_{n_\delta}) \leq \phi_\alpha(x - c_\alpha t_{n_\delta}) + \delta$$

for any $x \in \mathbb{R}$ (after a suitable translation).

Third Phase

Set

$$v(y,t) := u(y + c_\alpha t, t) - a_+ \in [\alpha - a_+, 0], \quad y \in \mathbb{R}_+, \quad t > t_2.$$

Then $(\mathbf{AC})_\alpha$ implies

$$\partial_t v - \partial_y^2 v + \eta + f(v + a_+) = c_\alpha \partial_y v, \quad \eta \in \partial I_{[\alpha,\infty)}(v + a_+)$$

in $\mathbb{R}_+ \times \mathbb{R}_+$. We can assume WLOG (by translation) that

$$v(0,t) = \alpha - a_+, \quad \partial_y v(0,t) = 0, \quad v(y,0) = u(y,0).$$

A key step is to establish local-energy estimates for $v(y,t)$.

Step 1 Based on [A-Efendiev '19], we can prove that

$$\sup_{t \geq 0} \int_{\mathbb{R}} |\eta(\cdot,t)|^2 \rho \, \mathrm{d}x \leq \int_{\mathbb{R}} |(\partial_x^2 u_0 - f(u_0))_-|^2 \rho \, \mathrm{d}x \quad \forall \rho \in C_c^\infty(\mathbb{R}).$$

Third Phase

Step 2 (Caccioppoli type estimate) Let $\zeta_R \in C_c^\infty(\mathbb{R})$ be such that

$$\zeta_R \equiv 1 \text{ in } [0,R], \quad \zeta_R \equiv 0 \text{ in } [2R,+\infty), \quad \|\zeta_R'\|_{L^\infty(\mathbb{R}_+)} \le \frac{2}{R}.$$

Test $(\mathbf{AC})_\alpha$ by $\mathrm{e}^{-\lambda t}\mathrm{e}^{c_\alpha y} v \zeta_R^2$. Then

$$\frac{1}{2}\mathrm{e}^{-2\lambda t}\int_0^{+\infty} \mathrm{e}^{c_\alpha y} v(\cdot,t)^2 \zeta_R^2 \,\mathrm{d}y + \frac{1}{2}\int_0^t \mathrm{e}^{-2\lambda\tau}\left(\int_0^{+\infty} \mathrm{e}^{c_\alpha y}|\partial_y v|^2 \zeta_R^2 \,\mathrm{d}y\right)\mathrm{d}\tau \le \frac{1}{2|c_\alpha|}\|v(\cdot,0)\|_{L^\infty(\mathbb{R}_+)}^2 + \frac{4}{\lambda |c_\alpha| R^2}\|v\|_{L^\infty(Q_T)}^2,$$

where we used the fact that $f'' \ge -\lambda$ and $v\eta \ge 0$. Let $R \to +\infty$. Then

$$\int_0^t \mathrm{e}^{-2\lambda\tau}\left(\int_0^{+\infty} \mathrm{e}^{c_\alpha y}|\partial_y v|^2 \,\mathrm{d}y\right)\mathrm{d}\tau \le C.$$

Third Phase

Step 3 (Weighted energy estimate for $\partial_t v$) Test $(\mathbf{AC})_\alpha$ by $e^{c_\alpha y}\partial_t v$. Then

$$\int_0^\infty e^{c_\alpha y}|\partial_t v|^2\,dy + \frac{d}{dt}\underbrace{\left(\frac{1}{2}\int_0^\infty e^{c_\alpha y}|\partial_y v|^2\,dy + \int_0^\infty e^{c_\alpha y}h(v)\,dy\right)}_{=E(v(\cdot,t))} = 0.$$

where $h(v) = \hat{f}(v + a_+) \geq 0$, since $\eta\partial_t v \equiv 0$ by $\eta \in \partial I_{[0,+\infty)}(\partial_t u)$. Integrate it over (τ_0, t) with $\tau_0 > 0$. It then follows from Step 1 that

$$\int_{\tau_0}^t \int_0^\infty e^{c_\alpha y}|\partial_t v(y,\tau)|^2\,dy d\tau + E(v(\cdot,t)) \leq E(v(\cdot,\tau_0)) \quad \forall t \geq \tau_0.$$

Step 4 (Quasi-convergence local in space) One can take $t_n \to \infty$ such that

$$\partial_t v(\cdot, t_n) \to 0 \quad \text{strongly in } L^2(\mathbb{R}_+; e^{c_\alpha y}dy),$$

Third Phase

$$\begin{aligned} \eta(\cdot, t_n) &\to \eta_\infty && \text{weakly in } L^2(0,R), \\ v(\cdot, t_n) &\to \psi && \text{weakly in } H^2(0,R), \\ & && \text{strongly in } C^1([0,R]) \end{aligned}$$

for some $\psi \in H^2_{\mathrm{loc}}(\mathbb{R}_+)$ and any $R > 0$. Set

$$\phi(y) = \begin{cases} \psi(y) + a_+ & \text{if } \; y \geq 0, \\ \alpha & \text{if } \; y < 0, \end{cases}$$

which then solves

$$-\phi'' + f(\phi) + I_{[\alpha,\infty)}(\phi) \ni c_\alpha \phi' \; \text{ in } \mathbb{R}, \quad \phi(0) = \alpha, \quad \phi'(0) = 0.$$

Claim: $\exists h_1 \in \mathbb{R}$ such that $\phi(x) = \phi_\alpha(x - h_1) \; \forall x \in \mathbb{R}$.

Third Phase

Thus there exists a sequence $t_n \to +\infty$ such that

$$\sup_{y \leq R} |u(y + c_\alpha t_n, t_n) - \phi_\alpha(y - h_1)| \to 0 \quad \text{for} \;\; R > 0.$$

Step 5 (Quasi-convergence global in space)

Recall U_γ and ϕ_α and use comparison argument to

$$|u(y+c_\alpha t_n, t_n) - \phi_\alpha(y-h_1)| \leq a_+ - \min\{u(y+c_\alpha t_n, t_n), \phi_\alpha(y-h_1)\}.$$

Final Phase

♠ Final Phase Modifying the argument in [X. Chen '92], we shall prove exponential convergence of $u(\cdot,t)$ to $\phi_\alpha(\cdot - c_\alpha t)$ over $\mathbb{R}$ as $t \to +\infty$ (without taking any subsequence). Set

$$w^{\pm}(x,t) := \phi_\alpha(x - c_\alpha t \pm \sigma\delta(1-\mathrm{e}^{-\beta t}) - h^{\pm}) \pm \delta \mathrm{e}^{-\beta t}$$

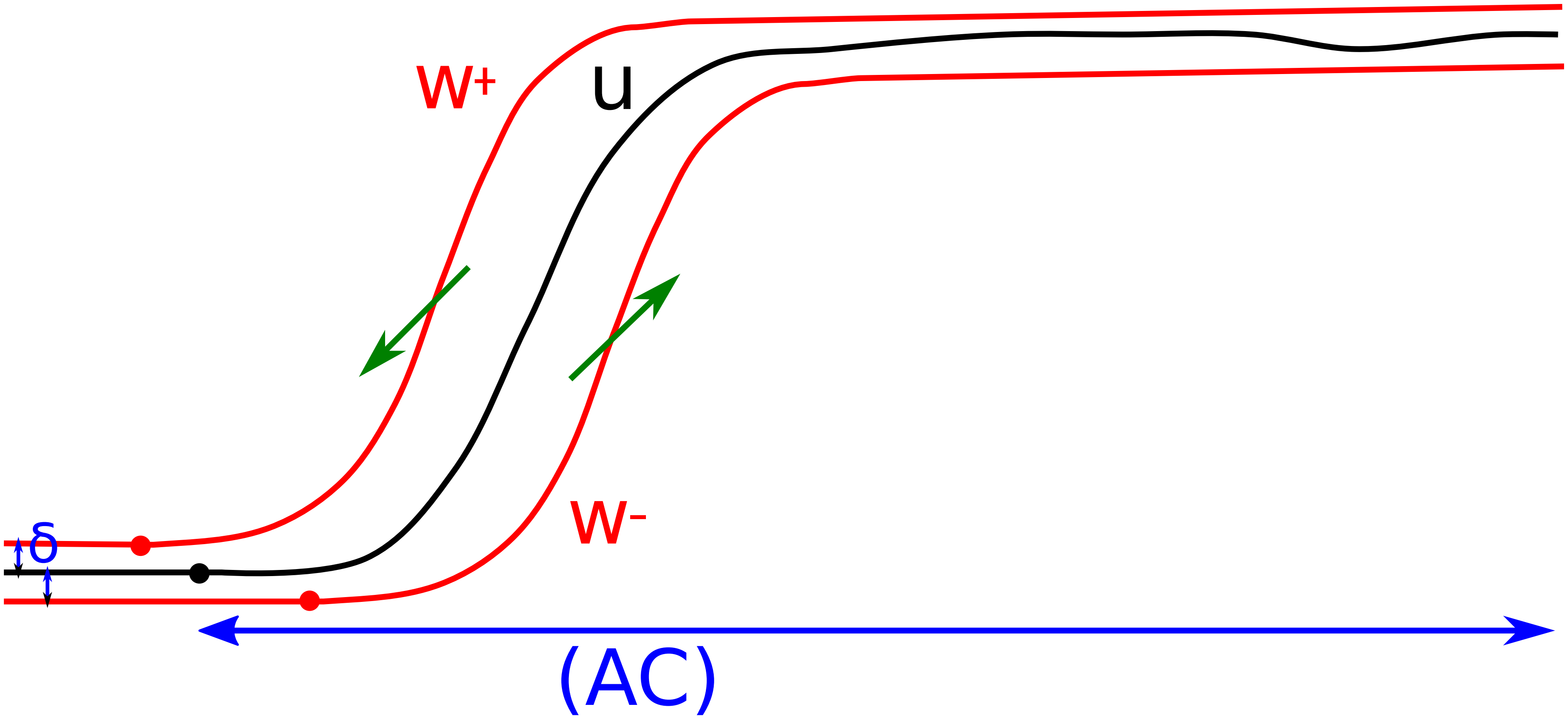
for $h^{\pm} \in \mathbb{R}$, $\delta \in (0,\delta_0)$ and $\beta, \sigma > 0$. Then $w^{\pm}$ turns out to be a super- and a subsolution to **(AC)**$_\alpha$, provided δ_0, β, σ are small enough.

Lemma 7 (Enclosing)

Set $h^+ = 0$ and assume $\phi_\alpha(\cdot - h^-) - \delta \le u(\cdot,0) \le \phi_\alpha(\cdot) + \delta$ in $\mathbb{R}$. If $\delta \in (0,\delta_0)$ and $h^- \ge 0$ are small enough, one can take $\varepsilon \in (0,1)$ and $t \gg 1$ such that

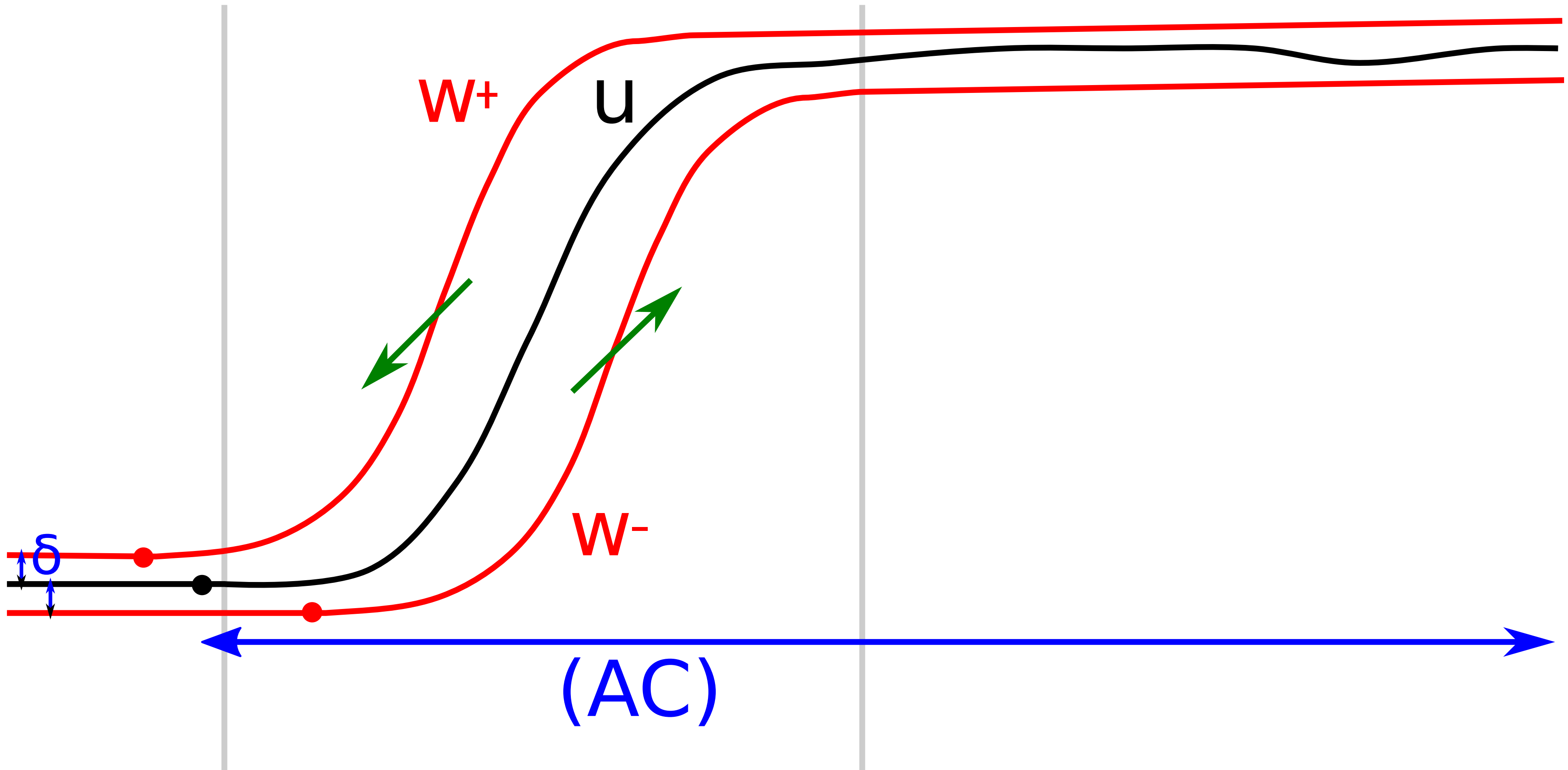
$$\phi_\alpha(x - c_\alpha t - \varepsilon(h^- - \delta)) - \varepsilon\delta \le u(x,t) \le \phi_\alpha(x - c_\alpha t + \varepsilon\delta) + \varepsilon\delta.$$

Final Phase



By comparison principle, $w^-(x,t) \leq u(x,t) \leq w^+(x,t)$.

Final Phase

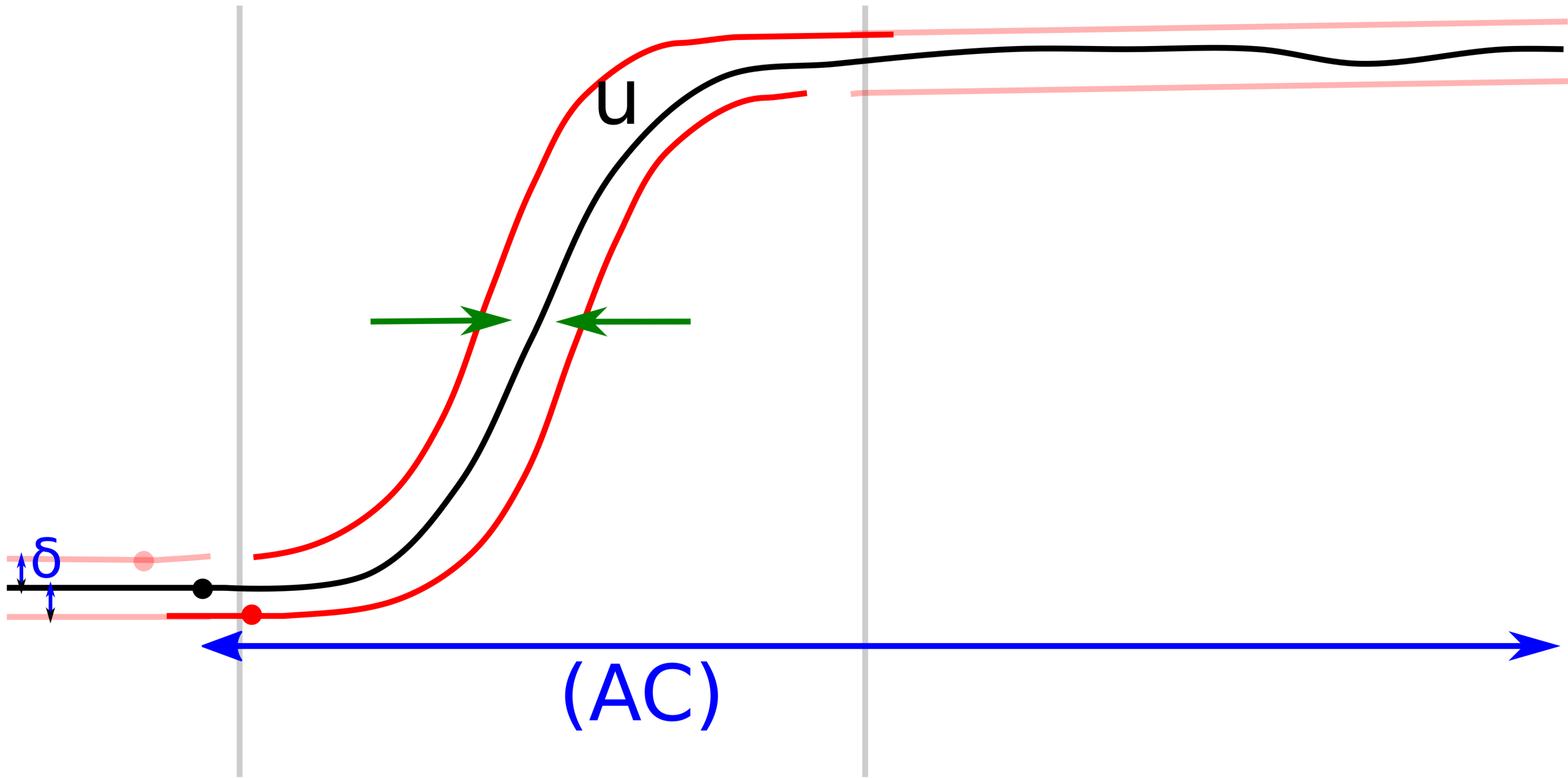


Set $W^{\pm} := \pm(w^{\pm} - u) \geq 0$. Then

$$\partial_t W^{\pm} - \partial_x^2 W^{\pm} \geq \mp \left(f(w^{\pm}) - f(u)\right) \geq -MW^{\pm}.$$

By strong maximum principle, one has $W^{+} \geq {}^{\exists}c_0 > 0$ or $W^{-} \geq c_0 > 0$.

Final Phase

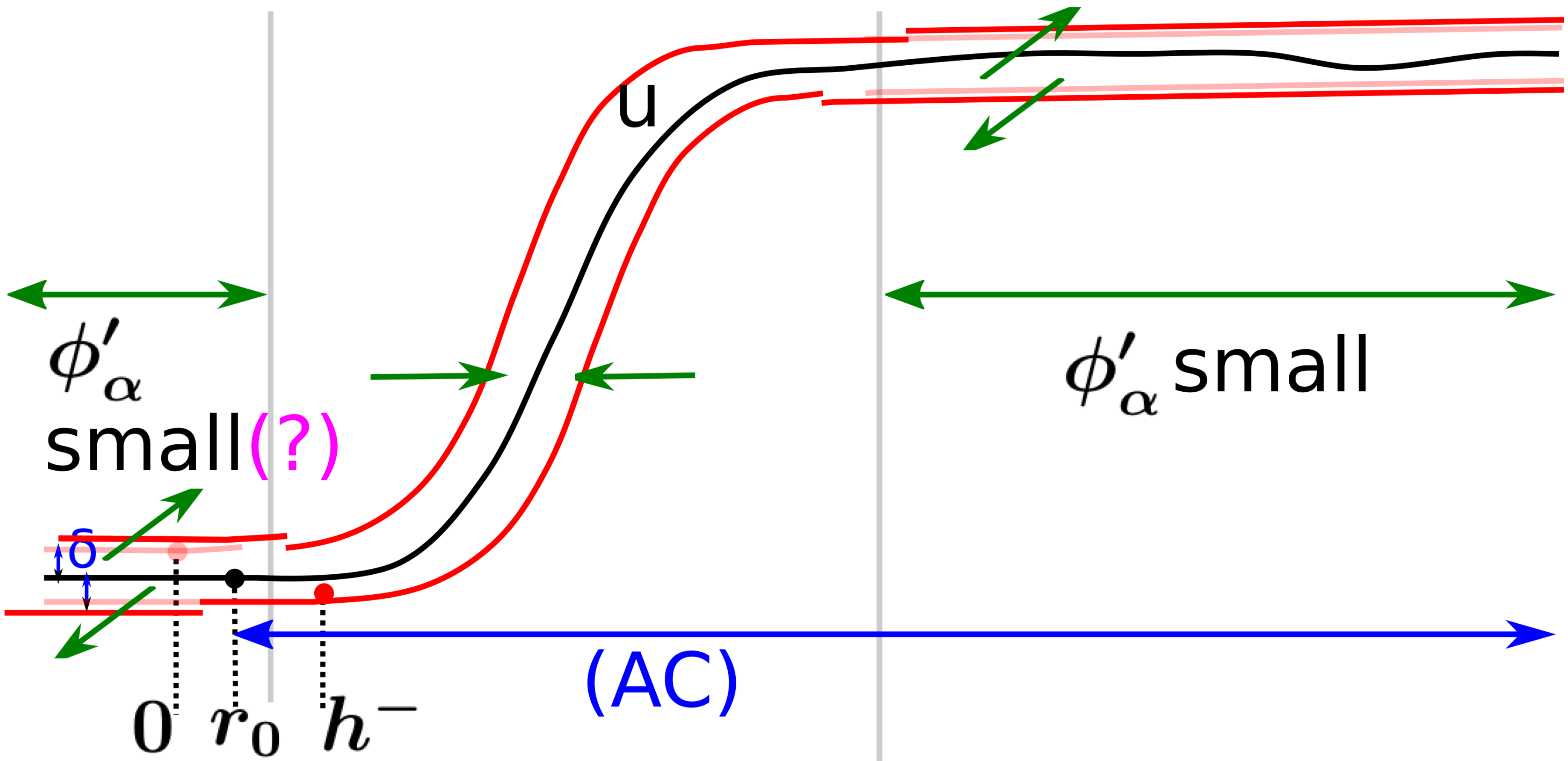


On the intermediate region, $W^+ \geq c_0 > 0$ implies

$$u(x,t) \leq \phi_\alpha(x - c_\alpha t + \sigma\delta(1 - e^{-\beta t}) - \Delta) + \delta e^{-\beta t}$$

for some $\Delta > 0$ ($\leftarrow$ improvement !).

Final Phase



If $\phi'_\alpha \ll 1$, one can then enclose $u(x,t)$ with smaller error:

$$\phi_\alpha(x - c_\alpha t) \approx \phi_\alpha(x - c_\alpha t - \Delta) - \Delta\, \phi'_\alpha(x - c_\alpha t)$$

On the left region, $0 < r_0 < h^-$ should be small enough.

Final remark

For sufficiently regular solutions, one can derive a motion equation for the free boundary $r(t)$ as follows:

$$\frac{\mathrm{d}r}{\mathrm{d}t}(t) = -\frac{\partial_x^3 u(r(t),t)}{f(\alpha)} \quad \text{for} \ \ t > 0.$$

cf.) Stefan problem

$$\frac{\mathrm{d}r}{\mathrm{d}t}(t) = -\mu \partial_x u(r(t),t) \quad \text{for} \ \ t > 0$$

[Du-Lin'10][Du-Guo'11,'12][Du-Lou-Zhou'15][Du-Matsuzawa-Zhou'15][Kaneko-Yamada'11,'18][Kaneko-Matsuzawa'15,'18][Kaneko-Matsuzawa-Yamada'20]...

cf.) The solution $u(x,t)$ turns out to satisfy

$$u(r(t),t) = \alpha, \quad \partial_x u(r(t),t) = 0, \quad \partial_x^2 u(r(t)+0,t) = f(\alpha)$$

for any $t > 0$.

Thank you for your attention !

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