Motion by mean curvature and coupled KPZ from particle systems

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MMC part: arXiv:2004.05276, Coupled KPZ part: arXiv:1908.07863

Outline

Goal: Interacting particle system

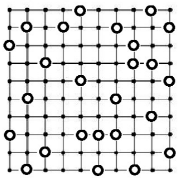
$$\xrightarrow[\text{scaling limits}]{} \text{ Nonlinear PDEs, Stochastic PDEs}$$

(e.g., Independent RWs $\to$ Linear heat equation $\partial_t u = \Delta u$)

Microscopic system

(=Interacting Random Walks with Creation and Annihilation)

- Glauber-Zero range process on large d-dim periodic lattice of size N



$\longleftarrow \quad N \quad \longrightarrow$

$d = 2$

$\circ$ = sites occupied by (several) particles

each particle jumps to neighboring sites

- Zero range part = Random walks interacting at same sites
- Glauber part = Creation and annihilation of particles with interaction

Micro $\to$ Macro: Scaling in Space and Time

- Zero range part (=Interacting RWs) produces nonlinear Laplacian
- Glauber part produces reaction term

- (A) (with Sethuraman, Hilhorst, El Kettani, Park)
  Glauber-Zero range on $\mathbb{T}_N^d \to$ Motion by mean curvature on $\mathbb{T}^d$

  $\mathbb{T}_N^d = \{1, 2, \ldots, N\}^d =$ d-dim discrete torus of size N: Micro
  $\mathbb{T}^d = [0, 1)^d =$ continuous torus of size 1: Macro

- (B) (with Bernardin, Sethuraman) Nonlinear Fluctuation:
  Multi-species Zero range process on $\mathbb{T}_N^1$ (No Glauber part)
  $\to$ Coupled KPZ equation (ill-posed SPDE) on $\mathbb{T}^1$

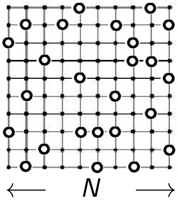
- [Keywords]
  - Hydrodynamic limit (local ensemble average via local ergodicity)
  - (1st and 2nd order) Boltzmann-Gibbs principle
  - nonlinear Allen-Cahn equation, sharp interface limit
  - ill-posed SPDE, renormalization

Part A

- Derivation of interface motion (Motion by mean curvature) directly from micro system called Glauber-Zero range process (=Interacting RWs with creation and annihilation)
- Proof: Combination of techniques of
  (1) Probabilistic method (called Hydrodynamic limit):
  Relative entropy method + Boltzmann-Gibbs principle
  (2) PDE method:
  Sharp interface limit for nonlinear Allen-Cahn equation
- For (2):
  Expansion up to 2nd order (corrector in homogenization theory)
  + Comparison theorem for discrete Allen-Cahn equation

1 Glauber-Zero range process

- Particles move on $\mathbb{T}_N^d = \{1, 2, \ldots, N\}^d$: discrete torus of size N
- Zero range process on $\mathbb{T}_N^d$:
  - ▷ $g(k)$ = Jump rate of one particle to one of neighboring sites when k particles exist at the same site
  - ▷ $g(k) = k \Leftrightarrow$ independent RWs, i.e., each particle has same jump rate 1. This produces linear Laplacian Δ at macroscopic level.
  - ▷ nonlinear $g(k)$ produces nonlinear Laplacian.
- Configuration: $\eta = \{\eta_x\}_{x \in \mathbb{T}_N^d} \in \mathcal{X}_N = \mathbb{Z}_+^{\mathbb{T}_N^d}$



$$\eta_x = \begin{cases} k, & k \text{ particles at } x \\ 0, & \text{no particle at } x \in \mathbb{T}_N^d \end{cases}$$
$$= \sharp\{\text{particles at } x\}$$

- "Ensemble" for Zero range part,
  i.e., Equilibrium (or Invariant) measures:
  - Translation-invariant product measures $\nu_\rho, \rho \in [0, \infty)$ with mean ρ (particle density) on configuration space $\mathcal{X}_N = \mathbb{Z}_+^{\mathbb{T}_N^d}$ (or $\mathcal{X} = \mathbb{Z}_+^{\mathbb{Z}^d}$).
- Glauber part: When the configuration is η,
  - $c_x^+(\eta)$ = Creation rate of one particle at x
    $c_x^-(\eta)$ = Annihilation rate of one particle at x
  - These rates are local functions and translation invariant: $c_x^\pm(\eta) = c^\pm(\tau_{-x}\eta)$, where τ is shift operator.
- Given the rates $g, c^\pm$ and initial configuration $\eta(0)$, we can construct the time evolution of particles on $\mathbb{T}_N^d$:

$$\eta(t) = \{\eta_x(t)\}_{x \in \mathbb{T}_N^d}, \quad t \geq 0$$

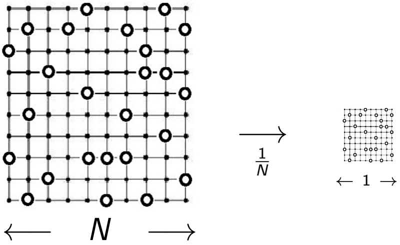
  called Glauber-Zero range process.
- We assume some conditions on $g, c^\pm$ and initial distribution of $\eta(0)$. (e.g., spectral gap for Zero range generator.)

Scaling from Micro to Macro

- The macroscopic empirical measure (density of particles) on $\mathbb{T}^d (= [0,1)^d$: macroscopic region) associated with configuration $\eta \in \mathcal{X}_N$ is defined by

$$\alpha^N(dv;\eta) = \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \eta_x \delta_{\frac{x}{N}}(dv), \quad v \in \mathbb{T}^d.$$

- space $\frac{1}{N}$, mass $\frac{1}{N^d}$



micro macro

- We also introduce time change for $\eta(t)$ s.t. time N^2 (for Zero range part) and time $K = K(N) \nearrow \infty$ (for Glauber part)

$$\rightarrow \quad \eta^N(t) = \{\eta_x^N(t)\}_{x \in \mathbb{T}_N^d}$$

2 Allen-Cahn equation at intermediate level

HD limit

- (Goal) We derive the homogenized motion by mean curvature (MMC) from our particle system.

- For K fixed, in the hydrodynamic limit, we have

$$\alpha^N(dv; \eta^N(t)) \to \rho(t,v)dv \quad \text{as } N \to \infty$$

  and obtain, by the local ergodicity leading to local ensemble averages, reaction-diffusion equation for the limit density $\rho = \rho^K$:

$$\partial_t \rho = \Delta \varphi(\rho) + Kf(\rho) \quad \text{on} \quad \mathbb{T}^d, \tag{1}$$

- Here

$$\varphi(\rho) = E^{\nu_\rho}[g(\eta_0)],$$
$$f(\rho) = E^{\nu_\rho}\left[c^+(\eta) - c^-(\eta)\right]$$

  are averages under ensembles ν_ρ of particle density ρ.

Choice of $c^{\pm}(\eta)$

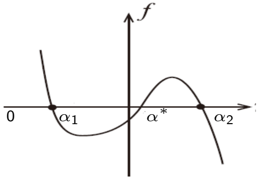
- One can construct creation/annihilation rates $c^{\pm}(\eta)$ such that the corresponding f is bistable:

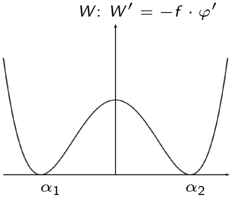
  f has exactly three zeros $0 < \alpha_1 < \alpha^* < \alpha_2$

  and $f'(\alpha_1) < 0, f'(\alpha^*) > 0, f'(\alpha_2) < 0$

  and satisfies φ-balance condition:

  $\int_{\alpha_1}^{\alpha_2} f(\rho)\varphi'(\rho)d\rho = 0.$
- The equation (1) is called (nonlinear) Allen-Cahn equation.
- One can derive (homogenized) MMC from A-C eq (1) as $K \to \infty$. (This PDE result looks new to the best of our knowledge.)





φ-Modified Potential

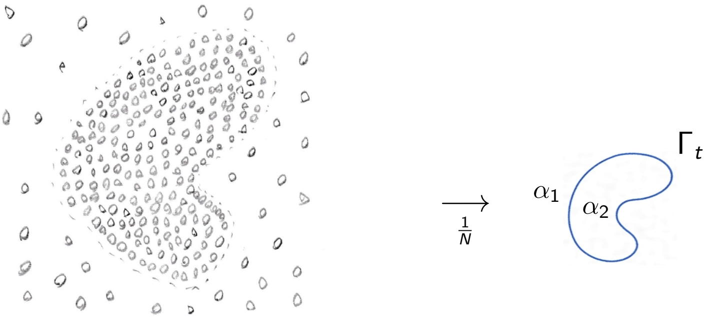
Phase separation

- Microscopically, the model has two different phases:

  Sparse region (roughly, density α_1) and

  Dense region (density α_2).

  Macroscopically, these two regions are separated by an interface Γ_t.
- Creation/annihilation mechanism at microscopic level keeps macroscopic density at each of these stable states under time evolution.



3 Main result of Part A

- $\alpha^N(t, dv) := \alpha^N(dv; \eta^N(t))$ is the corresponding macroscopic empirical measure.

Theorem 1: Assume several conditions on rates $g, c^{\pm}$, initial value $\eta^N(0)$ (e.g., $\alpha^N(0) \to \chi_{\Gamma_0}$ and entropy condition stated below) and $K(N) \to \infty, K(N) \le \delta(\log\log N)^{1/2}$ with small $\delta = \delta_T > 0$.
Then, we have for $t \in [0, T]$

$$\alpha^N(t) \to \chi_{\Gamma_t} := \begin{cases} \alpha_1, & \text{one side of } \Gamma_t, \\ \alpha_2, & \text{another side of } \Gamma_t, \end{cases}$$

in probability, where the sides are determined by Γ_0 and the hypersurface Γ_t in $\mathbb{T}^d$ moves according to the homogenized motion by mean curvature: $V = \lambda_0 \kappa$.

- V = normal velocity of Γ_t
- κ = mean curvature $\times(d-1)$ of Γ_t
- λ_0 = product of surface tension and mobility (next page)
- $T > 0$ is taken such that Γ_t is smooth for $t \in [0, T]$.

- (Entropy condition) $H(\mu_0^N|\nu_0^N) = O(N^{d-\delta_0})$ for some $\delta_0 > 0$, where μ_0^N is distribution (=probability law) of $\eta^N(0)$ and ν_0^N is product measure with mean $\{u^N(0,x)\}_x$ (i.e., ν_0^N = local equilibrium).
- $H(\mu|\nu)$ is the relative entropy:

$$H(\mu|\nu) := \int \frac{d\mu}{d\nu} \log \frac{d\mu}{d\nu} \cdot d\nu.$$

- λ_0, interpreted as the product of surface tension and mobility, is determined by homogenization effect from nonlinear Laplacian:

$$\lambda_0 = \frac{\int_{\mathbb{R}} \{\varphi'(U_0(z))U_0'(z)\}^2 dz}{\int_{\mathbb{R}} \varphi'(U_0(z))\{U_0'(z)\}^2 dz}$$

  where U_0 is the traveling wave solution for (1) with $K = 1$ connecting α_1 and α_2.
- $\lambda_0 = 1$ if $\varphi(u) = u$.

4 Proof of Theorem 1 (i.e., $\alpha^N(t) \to \chi_{\Gamma_t}$)

- Combination of probabilistic and PDE methods

4.1 Probabilistic part

- Let μ_t^N be the distribution (=probability law) of $\eta^N(t)$ on $\mathcal{X}_N$.

Choice of local equilibrium sates ν_t^N

- We choose ν_t^N appropriately as follows.
- Let $u^N(t) = \{u^N(t, \frac{x}{N})\}_{x \in \mathbb{T}_N^d}$ be the solution of the discrete hydrodynamic equation (discrete Allen-Cahn equation):

$$\partial_t u^N(t, \tfrac{x}{N}) = \Delta^N \varphi(u^N(t, \tfrac{x}{N})) + Kf(u^N(t, \tfrac{x}{N})). \qquad (2)$$

- Let $\nu_t^N \equiv \nu_{u^N(t)}$ be the product measure on $\mathcal{X}_N$ with mean $\{u^N(t, \frac{x}{N})\}_{x \in \mathbb{T}_N^d}$. This is a local equilibrium state with density determined by the discrete HD equation (2).

Theorem 2 (Main result in probabilistic part):
If $H(\mu_0^N|\nu_0^N) = O(N^{d-\delta_0})$ for some $\delta_0 > 0$, and if $1 \le K(N) \le \delta(\log\log N)^{1/2}$ for small $\delta > 0$, we have

$$H(\mu_t^N|\nu_t^N) = o(N^d).$$

- Once Theorem 2 is shown, by the entropy inequality + Large deviation estimate for ν_t^N, one can show that $\alpha^N(t)$ is close to $u^N(t)$, the solution of the discrete HD eq (2) with diverging factor $K(N)$. The limit of $u^N(t)$ is studied in PDE part.

Proof of Theorem 2

(i) H.-T. Yau's relative entropy method (to compute $\partial_t H(\mu_t^N|\nu_t^N)$ and see it is an average under μ_t^N of certain microscopic function of the form $h - \tilde{h}(u_{t,x}^N)$ with $\tilde{h}(\rho) = E^{\nu_\rho}[h]$)

(ii) Boltzmann-Gibbs principle: replacement of $h - \tilde{h}$ (under space-time average and μ_t^N) by linear fts of η_x with entropy and $o(N^d)$ errors.

(iii) Linear fts vanish if $u^N(t)$ is determined by discrete HD eq (2).

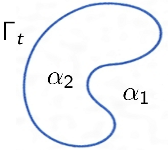
(iv) To control prefactor in h, we need the condition on $K(N)$.

(v) We finally apply Gronwall's inequality.

4.2 PDE part (Homogenization +Comparison argument)

Theorem 3: Under our assumptions, as $N \to \infty$

$$u^N(t) \to \chi_{\Gamma_t} := \begin{cases} \alpha_1, & \text{one side of } \Gamma_t, \\ \alpha_2, & \text{another side of } \Gamma_t, \end{cases}$$



where the hypersurface Γ_t in $\mathbb{T}^d$ moves according to the homogenized motion by mean curvature: $V = \lambda_0 \kappa$.

- Combining Theorems 2 and 3, Theorem 1: "$\alpha^N(t) \to \chi_{\Gamma_t}$ in probability" is shown.

Proof of Theorem 3

- We use comparison theorem for discrete PDE:
  If $u^{\pm}$ are super/sub solutions of (2) (i.e., they satisfy it in inequalities $\geq / \leq$) and $u^-(0) \leq u^+(0)$, then $u^-(t) \leq u^+(t)$.
- This follows from the non-decreasing property of φ.

Construction of super/sub solutions (with correctors):
— Propagation of interface

- We define $u^{\pm}(t,v), v \in \mathbb{T}^d$ by

$$\begin{aligned} u^{\pm}(t,v) =& U_0\big(K^{1/2}d(t,v) \pm p(t)\big) \\ &+ K^{-1/2}U_1\big(t,v,K^{1/2}d(t,v) \pm p(t)\big) \pm q(t). \end{aligned}$$

- Here $U_0 = U_0(z), z \in \mathbb{R}$ is a traveling wave solution connecting α_1 and α_2 for (1) with $K = 1$ on $\mathbb{R}$ and $d(t,v)$ is defined from the signed distance from Γ_t.
- Corrector: $U_1 = U_1(t,v,z)$ is the second term in the asymptotic expansion in K for the PDE (1):

$$\partial_t u = \Delta\varphi(u) + Kf(u).$$

- $p(t) = e^{-\beta tK} - e^{M_1 t} - M_2, q(t) = \sigma\left(\beta e^{-\beta tK} + \frac{M_1}{K}e^{M_1 t}\right),$ with properly chosen $\beta, \sigma, M_1, M_2 > 0$.

- Applying the comparison theorem for discrete PDE, we have

Proposition 4: Assume $\Gamma_t, t \in [0,T]$ is smooth and $K = o(N^{2/3})$ for $K = K(N) \to \infty$. Then, there exists $N_0 \in \mathbb{N}$ such that

$$u^-(t,v) \le u^N(t,v) \le u^+(t,v), \quad t \in [0,T], v = \frac{x}{N}, x \in \mathbb{T}_N^d$$

holds for every $N \ge N_0$.

- Initial layer problem (generation of interface) is also solved.
- By Proposition 4, one can complete the proof of Theorem 3.

Related results (Kawasaki=RWs with hard core exclusion)

- MMC from Glauber-Kawasaki dynamics: F-Tsunoda (JSP, 2019)
- Stefan problem from two component Glauber-Kawasaki dynamics: De Masi-F-Presutti-Vares (ALEA, 2019)

Part B

1. Multi-component coupled KPZ equation
2. n-species zero-range processes on $\mathbb{T}_N$
3. Nonlinear fluctuation leading to coupled KPZ equation

1. Multi-component coupled KPZ equation

- $\mathbb{R}^n$-valued KPZ eq for $h(t,u) = (h^i(t,u))_{i=1}^n$ on $\mathbb{T} = [0,1)$ (or $\mathbb{R}$):

$$\partial_t h^i = \tfrac{1}{2}\partial_u^2 h^i + \tfrac{1}{2}\Gamma^i_{jk}\partial_u h^j \partial_u h^k + \xi^i, \quad 1 \le i \le n.$$

- We use Einstein's convention.
- $\xi(t,u) = (\xi^i(t,u))_{i=1}^n \big(\equiv \dot{W}(t,u) \big)$ is an $\mathbb{R}^n$-valued space-time Gaussian white noise with covariance structure

$$E[\xi^i(t,u)\xi^j(s,v)] = \delta^{ij}\delta(u-v)\delta(t-s).$$

- We can generalize $\partial_u^2 h^i \to D^i_j \partial_u^2 h^j$ (cross diffusion system, D: symmetric, $D > 0$) and $\xi^i \to \sigma^i_j \xi^j$ (with another diffusion coefficient σ).
- The coupling constants Γ^i_{jk} always satisfy bilinear condition: $\Gamma^i_{jk} = \Gamma^i_{kj}$ for all i,j,k, and (sometimes) trilinear condition

$$\Gamma^i_{jk} = \Gamma^i_{kj} = \Gamma^j_{ik} \text{ for all } i,j,k. \qquad \textbf{(T)}$$

- $\Gamma = 0 \Rightarrow h \in C^{\frac{1}{4}-,\frac{1}{2}-}([0,\infty) \times \mathbb{T})$ a.s.
- In general with Γ, $\partial_u h^i \in C^{-\frac{1}{2}-}(\mathbb{T})$ so that KPZ equation is ill-posed. $\to$ Hairer's theory of regularity structure '14
- Role of trilinear condition (T): drop noise and compute $\partial_t \|\partial_u h\|^2_{L^2(\mathbb{T})}$, then we have the term

$$\sum_{i,j,k} \Gamma^i_{jk} \int_{\mathbb{T}} \partial_u h^j \partial_u h^k \partial_u^2 h^i \, du$$

  and this $= 0$ under (T) by integration by parts. "Converse" is also true, i.e., if this vanishes for wide class of h, then (T) holds.
- This property is similar to Euler/Navier-Stokes equations.

Results on coupled KPZ equation (F-Hoshino JFA '17 on $\mathbb{T}$)

- Local solvability with renormalization by applying paracontrolled calculus due to Gubinelli-Imkeller-Perkowski '15.
- Under the trilinear condition (T),
  - (unique) invariant measure = Wiener measure
  - Global existence, uniqueness for all initial values in $\mathcal{C}^\alpha, \alpha < \frac{1}{2}$
  - cancellation in log-renormalization (for 4th order terms)
  - two types of approximations, difference of two limits (cf. F-Quastel '15 when $n = 1$)

Motivation to study coupled KPZ eq:
Nonlinear fluctuating hydrodynamics (Spohn)

Our goal: Derivation of coupled KPZ equation from microscopic systems.

When $n = 1$ (single component scalar-valued case), this was done by Bertini-Giacomin (Cole-Hopf solution), Goncalves-Jara, Goncalves-Jara-Sethuraman.

2. n-species zero-range processes on $\mathbb{T}_N$

- To derive n-component system in the limit, we need to consider a system with n-conserved quantities (n-species) at microscopic level.
- $\mathbb{T}_N = \{1, 2, \ldots, N\}$ with periodic boundary condition. This is a microscopic space corresponding to macroscopic $\mathbb{T} = [0, 1)$.
- Configuration space of particles: $\eta = (\eta^i)_{i=1}^n \in \Omega^n$, $\Omega = \mathbb{Z}_+^{\mathbb{T}_N}$.
- $\eta_x^i \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, $x \in \mathbb{T}_N$, $1 \le i \le n$:
  number of ith species particles at x.
- (Grosskinsky-Spohn) Jump rate $g_i(\eta_x)$ of ith species particle depends on $\eta_x = (\eta_x^i)_{i=1}^n$ (only on numbers of particles at x) and satisfies the compatibility condition $\to$ Detailed balance w.r.t. product measures (ensembles) $\{\nu_a; a \in [0, \infty)^n\}$.
- Weak asymmetry: Once jump happens, the probability of jump to right is $\frac{1}{2} + \frac{c}{N^\gamma}$ and to left is $\frac{1}{2} - \frac{c}{N^\gamma}$, $c > 0$.
- We introduce a diffusive time change $t \mapsto N^2 t$ for the microscopic process. The process is denoted by $\eta^N(t) = (\eta_x^{N,i}(t))_{x \in \mathbb{T}_N^d, 1 \le i \le n}$.
- $\gamma = 1$ for HD limit
- $\gamma = \frac{1}{2}$ for KPZ fluctuation.

3. Nonlinear fluctuation leading to coupled KPZ equation

- We now take $\gamma = \frac{1}{2}$, i.e., $\frac{1}{2} + \frac{c}{\sqrt{N}}$ to right and $\frac{1}{2} - \frac{c}{\sqrt{N}}$ to left.
- We consider the fluctuation field under equilibrium, i.e. $\eta_0^N \overset{\text{law}}{=} \nu_{a_0}$.
- To cancel some diverging factor (drift in HD limit), we introduce the moving frame with speed $2c\lambda N^{\frac{3}{2}}$ at microscopic level with suitably chosen $\lambda = \lambda(a_0)$.

$$Y_t^{N,i}(du) := \frac{1}{\sqrt{N}} \sum_x \left(\eta_x^{N,i}(t) - a_0^i\right) \delta_{\frac{x}{N} - \frac{2c\lambda N^{3/2} t}{N}}(du)$$

- The frame should have common speed for all i.
  $\to$ This gives a restriction to the choice of a_0.

Main result of Part B

- We choose a_0 and $\lambda(a_0)$ properly.

Theorem 5: The limit $Y_t = (Y_t^i)_{i=1}^n$ of $Y_t^N = (Y_t^{N,i})_{i=1}^n$ is the equilibrium (unique energy) solution of coupled KPZ-Burgers equation:

$$\partial_t Y^i = \frac{1}{2} Q_i(a_0) \partial_u^2 Y^i + \Gamma_{jk}^i(a_0) \partial_u (Y^j Y^k) + q_i(a_0) \partial_u \xi^i$$

- Convergence is in law in the space $D([0,T], \mathcal{S}'(\mathbb{T})^n)$.
- Here $\xi = (\xi^i)_{i=1}^n$ are n independent space-time white noises.
- $Q_i(a)$, $\Gamma_{jk}^i(a)$ and $q_i(a)$ are given by

$$Q_i(a) = \partial_{a^i} \tilde{g}_i(a), \quad \Gamma_{jk}^i(a) = c \partial_{a^j} \partial_{a^k} \tilde{g}_i(a), \quad q_i(a) = \sqrt{\tilde{g}_i(a)}.$$

- $\tilde{g}_i(a)$ are ensemble averages of g_i under Bernoulli measures with mean $a = (a^i)_{i=1}^n$.
- We can also derive additional linear drift term $+c_i \partial_u Y^i$ by considering i-dependent weak asymmetry.

- h^i: coupled KPZ $\Longleftrightarrow$ $Y^i := \partial_u h^i$: coupled KPZ-Burgers

Proof

- For the proof, we need to establish the 2nd order Boltzmann-Gibbs principle, i.e., replacement under space-time average of nonlinear function f of η s.t. $\tilde{f}(a_0) = \partial_{a^i}\tilde{f}(a_0) = 0 \ ({}^\forall i)$ by quadratic function of $\eta^i - a^i$. We use equivalence of ensembles and spectral gap.
- For the identification of the limit, we use the uniqueness of stationary coupled energy solutions due to Gubinelli-Perkowski.
- stationary energy solution $=$ martingale solution$+$Yaglom reversibility $+$ L^2-energy condition (convergence of nonlinear term)
- At Burgers level, we don't see the renormalization.

Trilinear condition

- Our $\Gamma(a_0)$ satisfies the trilinear condition (T) after rewriting it in a canonical form by change of time and magnitude.
- At least heuristically,
  (T)$\Longleftrightarrow$ "invariant measure of coupled KPZ=white noise"

Summary of the talk

Part A (HD limit, LLN)

1. Derivation of interface motion from interacting particle systems with additional large factors
2. Combination of relative entropy method, Boltzmann-Gibbs principle and techniques of PDEs
3. Motion by mean curvature with homogenization effect from the nonlinear Laplacian

Part B (Nonlinear fluctuation)

1. Ill-posed system of SPDE from particle system
2. Renormalization, trilinear condition, 2nd order Boltzmann-Gibbs principle

Thank you for your attention!